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The Representation of Finite Rotations in Spatial Kinematics

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1. INTRODUCTION

1.1 The possible approaches to finite rotations

The objective of this first chapter is to describe the various possible approaches available in computational kinematics and dynamics to describe the arbitrarily large rotations undergone by rigid and flexible bodies, and to compute the associated angular velocities and accelerations. It has become a problem of considerable interest for new disciplines such as computer aided analysis of mechanisms and finite element analysis of large space structures.

We have identified mainly three complementary ways to describe large rotations, each one shedding its own light on the problem.

- The *geometrical approach* is the most classical one. It is the one usually found in most textbooks on dynamics. It leads to finite rotation representation tools such as Euler angles, Bryant (or nautical) angles, single rotation about an arbitrary axis in space (Euler–Chasles representation), Euler and Rodrigues parameters.
- The *matrix approach* makes use of the well known *orthonormality* property of a rotation operator to obtain its explicit algebraic structure. As it will be seen, one is led

in this way to a rotation operator expressed in terms of either Euler or Rodrigues parameters. In this context, the decomposition of the operator into its vector and scalar parts will be described. Here, we also present the derivation of other parametrization techniques, like the rotational vector, the conformal rotational vector and the linear parameters.

- The *algebraic approach*, which can be focused from two different points of view:

  quaternion algebra leads very directly, and in a extremely elegant manner, to the representation of finite rotations. Making use of the quaternion algebra allows at the same time to combine a sequence of finite rotations in a product of quaternions, leading thus to a minimum number of algebraic operations. It gives rise to a systematic and very performing matrix formalism.

  matrix algebra employs concepts borrowed from differential geometry and allows to establish simple relations between engineering measures of rotations and the employed system of parameters. Using matrix algebra, the linearization of equations is easily performed.

All approaches are fully described, and the principal relationships between them are established. Differentiability properties of each parameters system are analyzed. Final conclusions are drawn comparing the different parametrizations from various points of view, allowing us to make a convenient choice to be used in a mechanisms analysis program.

In order to introduce the concept of finite rotation operator, we will first recall as an introduction how to describe a frame transformation and how to compute the velocities and accelerations associated to the transformation.

1.2 Body fixed to reference frame transformation

Let P be an arbitrary point on a body B. Let us denote by (x_1, x_2, x_3) its coordinates in an absolute reference (orthonormal) frame $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, and by (X_1, X_2, X_3) its coordinates in a relative (orthonormal) frame $\{O'; \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ rigidly fixed to body B. In vector notation, the position vector at point P can be decomposed in the form

$$\vec{x} = \vec{x}_0 + \vec{X} \tag{1.1}$$

where

$$\vec{x} = \vec{OP}, \qquad \vec{X} = \vec{O'P} \qquad \text{and} \qquad \vec{x}_0 = \vec{OO'}$$

In order to express eqn (1.1) in matrix notation, let us associate to the vectors $\vec{x}$, $\vec{x}_0$

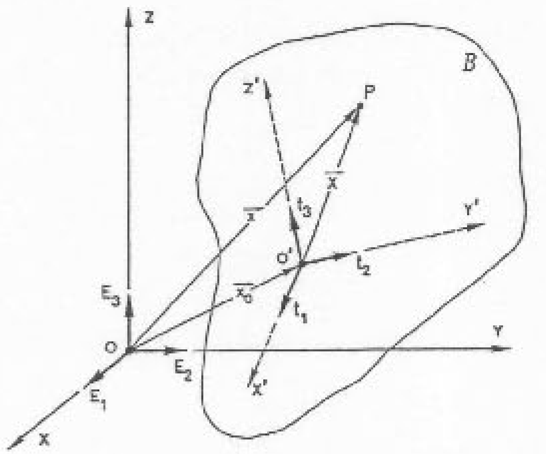

Figure 1: Body fixed to reference frame transformation

and $\vec{X}$ the unicolumn matrices

$$\begin{aligned} \mathbf{x}^T &= \langle\, x_1 \quad x_2 \quad x_3 \,\rangle \\ \mathbf{X}^T &= \langle\, X_1 \quad X_2 \quad X_3 \,\rangle \\ \mathbf{x}_0^T &= \langle\, x_{0\,1} \quad x_{0\,2} \quad x_{0\,3} \,\rangle \end{aligned}$$

The components $\mathbf{X}$ of vector $\vec{X}$ being expressed in the body reference frame $\{O'; \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$, the corresponding components $\mathbf{X}'$ in the inertial frame are obtained by a transformation

$$\mathbf{X}' = \mathcal{R}(\mathbf{X})$$

Lengths are unchanged under the transformation $\mathcal{R}$. Thus, it can be shown that $\mathcal{R}$ is an additive and homogeneous operator, which implies in turn that $\mathcal{R}$ is linear [1]. Then, we will represent it in the orthonormal basis $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ by means of the 3×3 matrix $\mathbf{R}$:

$$\mathbf{X}' = \mathbf{R}\,\mathbf{X} \tag{1.2}$$

$\mathbf{R}$ expresses the transformation from the frame $\{O'; \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ to $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$.

Let us express that the transformation (1.2) preserves the length of vector $\mathbf{X}$:

$$\mathbf{X}'^T\,\mathbf{X}' = \mathbf{X}^T\,\mathbf{R}^T\,\mathbf{R}\,\mathbf{X} = \mathbf{X}^T\,\mathbf{X} \tag{1.3}$$

This implies the algebraic condition on matrix $\mathbf{R}$

$$\mathbf{R}^T\,\mathbf{R} = \mathbf{1} \tag{1.4}$$

These matrices are called *unitary* or orthonormal matrices. They are classified in two categories, depending on the value of the determinant: those whose determinant is $+1$ are called proper orthogonal and those with determinant -1 are called *improper orthogonal.* Proper orthogonal matrices represent rigid body rotations, while improper orthogonal ones represent reflections [1].

Therefore, the matrix operation of the product of an arbitrary vector $\mathbf{X}$ by a proper orthonormal matrix $\mathbf{R}$ has the meaning of a rotation

$$\mathbf{X}' = \mathbf{R}\,\mathbf{X} \qquad \text{with} \qquad \mathbf{R}^{-1} = \mathbf{R}^T . \tag{1.5}$$

We note that by considering the body reference frame $\{O'; \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ coincident with the fixed inertial frame $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ at $t = 0$, one can see that $\mathbf{R}$ is also the transformation matrix for the rigid body rotation from the initial position to the actual one. In the following we will almost always refer to this signification.

From (1.5), one deduces the transformation

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{R}\,\mathbf{X} \tag{1.6}$$

which expresses in matrix form the vector relationship (1.1). It describes the position and orientation transformation resulting from a rigid body rotation and translation.

Let us note that the direct consequence of the orthonormality property of $\mathbf{R}$ is that it can be expressed in terms of 3 independent parameters. Indeed it is a 3×3 matrix, made of 9 elements, which can be written in terms of the column vectors

$$\mathbf{R} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3] \tag{1.7}$$

Owing to the orthonormality property (1.4) the vectors $\mathbf{r}_j$ are linked together by the 6 constraints

$$\mathbf{r}_i^T\,\mathbf{r}_j = \delta_{ij} \qquad (\mathrm{i} = 1,\ldots\mathrm{j}, \quad \mathrm{j} = 1, 2, 3) \tag{1.8}$$

Therefore we may write

$$\mathbf{R} = \mathbf{R}(\alpha_1, \alpha_2, \alpha_3) \tag{1.9}$$

where α_1, α_2, α_3 are three independent parameters retained to describe the rotation. Numerous choices exist, according to the representation technique adopted. These choices will be described in the next sections.

Euler theorem [1]

It can be shown that a proper orthogonal matrix has exactly one eigenvalue equal to $+1$. If $\mathbf{n}$ is the eigenvector of $\mathbf{R}$ corresponding to $+1$, it follows that

$$\mathbf{R}\,\mathbf{n} = \mathbf{n} \tag{1.10}$$

and furthermore, for any α

$$\mathbf{R}\,\alpha\,\mathbf{n} = \alpha\,\mathbf{n} \tag{1.11}$$

So, all points of the rigid body located along a line passing through a fixed point O and parallel to $\mathbf{n}$, remain fixed under the rotation $\mathbf{R}$. This line is called the *axis of rotation*.

The angle of rotation is measured on a plane perpendicular to the axis of rotation (see paragraph 2.7).

1.3 Translation and rotation velocities

Let us make the assumption that body B on fig. 1 is a rigid one. Let us restart from the matrix expression of the position vector of point P

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{R}\,\mathbf{X} \tag{1.12}$$

The components of the corresponding *velocity vector* are obtained through time differentiation

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \mathbf{R}\,\dot{\mathbf{X}} + \dot{\mathbf{R}}\,\mathbf{X} \tag{1.13}$$

where

$\dot{\mathbf{x}}_0$ represents the velocity vector at the reference point O'

$\dot{\mathbf{X}} = 0$ since body B is a rigid one.

It provides the expression for the velocity of an arbitrary point on the body

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \dot{\mathbf{R}}\,\mathbf{X} \tag{1.14}$$

Equation (1.14) can still be modified by expressing velocities in terms of quantities in the inertial reference frame. To this purpose, let us invert eqn (1.12) in the form

$$\mathbf{X} = \mathbf{R}^T\,(\mathbf{x} - \mathbf{x}_0) \tag{1.15}$$

This yields then to the velocity expression

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \dot{\mathbf{R}}\mathbf{R}^T\,(\mathbf{x} - \mathbf{x}_0) \tag{1.16}$$

It is possible to show that this expression for the absolute velocity at point P is the matrix form, giving extrinsic expression in terms of cartesian coordinates, of the vector relationship

$$\frac{d\vec{x}}{dt} = \frac{d\vec{x}_0}{dt} + \vec{\omega}\times(\vec{x} - \vec{x}_0) \tag{1.17}$$

where $\vec{\omega}$ is the *angular velocity vector* of frame $\{O'; \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ relative to $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$.

In order to check it, let us invoke the orthonormality property for $\mathbf{R}$:

$$\mathbf{R}\,\mathbf{R}^T = \mathbf{1} \quad\rightarrow\quad \dot{\mathbf{R}}\,\mathbf{R}^T + \mathbf{R}\,\dot{\mathbf{R}}^T = \mathbf{0} \tag{1.18}$$

from which follows that the matrix

$$\dot{\mathbf{R}}\,\mathbf{R}^T = -\mathbf{R}\,\dot{\mathbf{R}}^T$$

is skew-symmetric.

Let us then define the *matrix of angular velocities*

$$\tilde{\omega} = \dot{\mathbf{R}}\mathbf{R}^T = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{1.19}$$

where ω_1, ω_2, ω_3 are the cartesian components of vector $\vec{\omega}$ in the reference frame $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$.

With the definition (1.19), equation (1.16) can finally be rewritten in the form

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \tilde{\omega}\,(\mathbf{x} - \mathbf{x}_0) \tag{1.20}$$

which is the matrix analog of (1.17).

1.4 Translation and rotation accelerations

Let us keep the indeformability hypothesis for body B and restart from equation (1.12). A second time differentiation yields

$$\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_0 + \ddot{\mathbf{R}}\,\mathbf{X} \tag{1.21}$$

or, by taking account of (1.15)

$$\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_0 + \ddot{\mathbf{R}}\mathbf{R}^T(\mathbf{x} - \mathbf{x}_0) \tag{1.22}$$

The meaning of the matrix $\ddot{\mathbf{R}}\mathbf{R}^T$ can be obtained from a time differentiation of the angular velocity matrix $\tilde{\omega}$ given by eqn (1.19):

$$\frac{d\tilde{\omega}}{dt} = \tilde{\alpha} = \ddot{\mathbf{R}}\mathbf{R}^T + \dot{\mathbf{R}}\dot{\mathbf{R}}^T = \ddot{\mathbf{R}}\mathbf{R}^T + \tilde{\omega}\tilde{\omega}^T = \ddot{\mathbf{R}}\mathbf{R}^T - \tilde{\omega}^2 \tag{1.23}$$

where

$$\tilde{\alpha} = \dot{\tilde{\omega}} = \begin{bmatrix} 0 & -\dot{\omega}_3 & \dot{\omega}_2 \\ \dot{\omega}_3 & 0 & -\dot{\omega}_1 \\ -\dot{\omega}_2 & \dot{\omega}_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} \tag{1.24}$$

denotes the matrix of angular accelerations.

Substitution of (1.23) in (1.22) provides the expression for the acceleration at point P

$$\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_0 + (\tilde{\alpha} + \tilde{\omega}^2)\,(\mathbf{x} - \mathbf{x}_0) \tag{1.25}$$

where

- the first term represents the translational acceleration;

- the second one represents the contribution of angular acceleration;
- the third one represents the centrifugal acceleration contribution.

Equations (1.12), (1.20) and (1.25) together with the definitions (1.9), (1.19) and (1.24) provide the matrix notation for the kinematics of a rigid body.

1.5 Interpretation of arbitrary rigid body motion as screw motion

It is interesting to note that the differential motion of a rigid body as described by eqn (1.20) may be interpreted as *screw motion*, i.e. the combination of a translation and a rotation about a same axis $\mathbf{s}$ with arbitrary orientation in space.

To this purpose, let us determine the locus of points P having a velocity vector $\dot{\mathbf{x}}$ parallel to that of the angular velocity ω. The problem is to find a scalar quantity σ such that

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \tilde{\omega}\,(\mathbf{x} - \mathbf{x}_0) = \sigma\,\omega \tag{1.26}$$

Premultiplying eqn (1.26) by ω^T provides the expression

$$\sigma = \frac{1}{\omega^2}\,\omega^T\dot{\mathbf{x}}_0 \tag{1.27}$$

and the locus itself is obtained by solving (1.26) with respect to $\mathbf{x}$. To this purpose, let us rewrite the system of equations to be solved as follows

$$\tilde{\omega}\,\mathbf{x} = \frac{\omega^T\dot{\mathbf{x}}_0}{\omega^2}\,\omega - \dot{\mathbf{x}}_0 + \tilde{\omega}\,\mathbf{x}_0 = \mathbf{b} \tag{1.28}$$

By examining this system we note that

- obviously, a solution to (1.28) exists if and only if $\omega^T\mathbf{b} = 0$. It is easy to verify that this orthogonally condition is fulfilled here.
- a particular solution is obtained by seeking a solution in the form

$$\mathbf{x} = k\,\tilde{\omega}\,\mathbf{b} + \mu\,\omega \tag{1.29}$$

Its direct substitution in (1.28) gives

$$\tilde{\omega}\,\mathbf{x} = k\,\tilde{\omega}^2\,\mathbf{b} = k\,(\omega\omega^T - \omega^2\mathbf{1})\,\mathbf{b} = \mathbf{b} \tag{1.30}$$

After noticing that $\omega^T\mathbf{b} = 0$, we obtain

$$k = -\frac{1}{\omega^2} \tag{1.31}$$

The general solution to (1.28) is thus

$$\begin{aligned} \mathbf{x} &= -\frac{1}{\omega^2}\,\tilde{\omega}\,\mathbf{b} + \mu\,\omega = \mathbf{x}_0 - \frac{1}{\omega^2}\,(\omega^T \mathbf{x}_0)\,\omega - \frac{1}{\omega^2}\,\tilde{\mathbf{x}}_0\,\omega + \mu\,\omega \\ &= \mathbf{x}_0 + \mu\,\omega \end{aligned} \tag{1.32}$$

where the latter equality is obtained after using the arbitrariness of μ.

Equation (1.32) describes the locus of points having a velocity parallel to ω. It is a straight line s with direction ω and free parameter μ, passing through $\mathbf{x}_0$. Hence forth we will write

$$\mathbf{s} = \mathbf{x}_0 + \mu\,\omega \tag{1.33}$$

All points on s have, clearly, the same velocity $\dot{\mathbf{x}}_0 = \sigma\,\omega$ as can be verified by replacing (1.33) into (1.26).

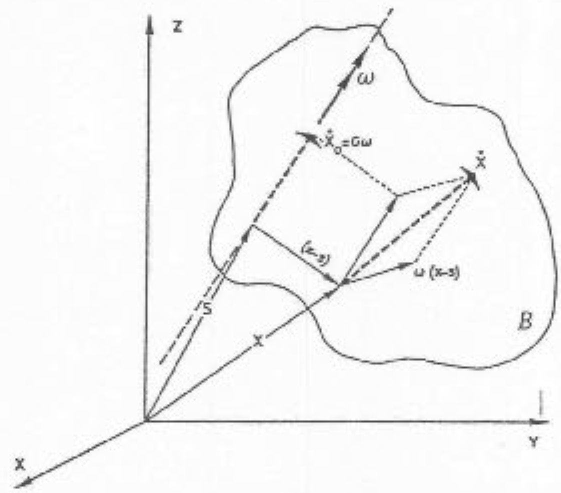

Figure 2: Interpretation of rigid body motion as screw motion

If we compute $\mathbf{x}_0$ from (1.33) and we replace it into (1.26), the following expression for the velocity at an arbitrary point P is obtained

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{\mathbf{x}}_0 + \tilde{\omega}\,(\mathbf{x} - \mathbf{s}) \\ &= \sigma\,\omega + \tilde{\omega}\,(\mathbf{x} - \mathbf{s}) \end{aligned} \tag{1.34}$$

It is easily seen that it corresponds to a screw motion characterized by

- a first component of translational velocity parallel to the rotation axis ω with a *pitch velocity* σ given by eqn (1.27);
- a second component resulting from angular motion with rotational velocity ω around the straight line s, equal to the angular speed cross product the distance from the point P to s. The straight line s can thus be interpreted as the *screw axis* of the helical motion.

This form of expressing motion of a rigid body is stated by Chasles theorem, which says that the most general displacement of a rigid body is equivalent to a translation together with a rotation about an axis parallel to the translation.

Some properties of the screw axis follow:

- The velocities of all the points of a rigid body undergoing an arbitrary motion have the same projection along the screw axis.
- The difference vector of the velocities of any two of the points of a rigid body undergoing an arbitrary motion is perpendicular to the screw axis.
- If the velocities of 3 noncollinear points of a rigid body are identical, the body undergoes a pure translation.

This properties can all be shown by applying equation (1.34), but they can also easily accepted by intuition. Ref.[1] gives a detailed demonstration of them.

2. GEOMETRIC DESCRIPTION OF A FINITE ROTATION OPERATOR

2.1 The plane rotation operator

The simplest rotation operation that can be considered is the rotation about a coordinate axis. Let us consider the case of fig. 3 where a rotation ϕ is performed about the z coordinate axis.

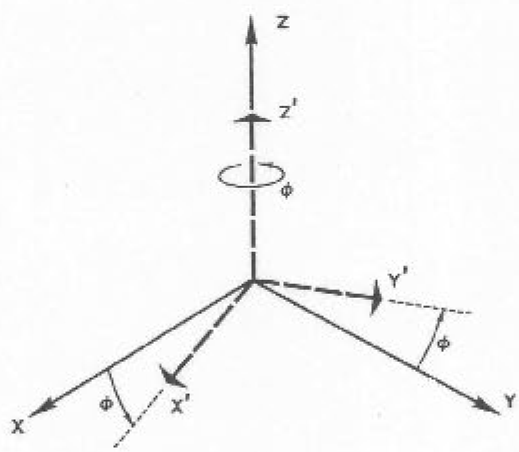

Figure 3: Rotation in plane Oxy

For a vector $\mathbf{x}$ with components $\langle x_1 \quad x_2 \quad x_3 \rangle$, one obtains the components of the rotated vector $\mathbf{x}'$

$$\begin{aligned} x'_1 &= x_1 \cos\phi \; - \; x_2 \sin\phi \\ x'_2 &= x_1 \sin\phi \; + \; x_2 \cos\phi \\ x'_3 &= x_3 \end{aligned}$$

In matrix form

$$\mathbf{x}' = \mathbf{R}\,\mathbf{x}$$

with the rotation operator

$$\mathbf{R}(z,\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

Similarly, for a rotation θ about the y axis and a rotation ψ about the x axis, one would obtain the rotation operators

$$\mathbf{R}(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.2}$$

and

$$\mathbf{R}(x,\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \tag{2.3}$$

2.2 Finite rotations in terms of direction cosines

The most obvious expression for a rotation operator performing an arbitrary rotation is the one obtained in terms of direction cosines. Let us denote (fig. 4) by $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ the unit vectors describing the cartesian frame $Oxyz$. Let also $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ be the basis vectors of the body frame $O'x'y'z'$, obtained by application of $\mathbf{R}$ to $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$:

$$\mathbf{t}_i = \mathbf{R}\ \mathbf{E}_i \tag{2.4}$$

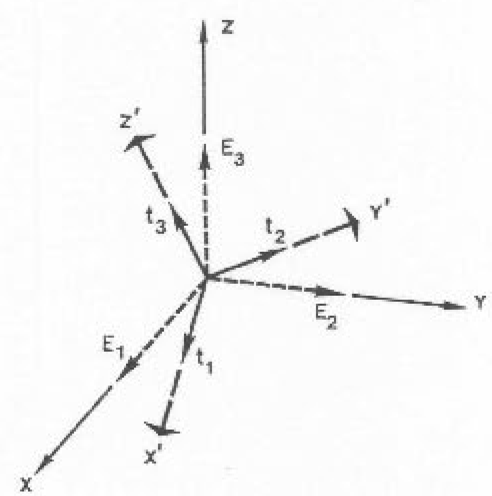

Figure 4: Finite rotation in terms of direction cosines

The rotated vector $\mathbf{x}'$ is obtained by applying $\mathbf{R}$ to $\mathbf{x}$

$$\mathbf{x}' = x'_i\,\mathbf{E}_i = \mathbf{R}\,\mathbf{x} = \mathbf{R}\,x_j\,\mathbf{E}_j = x_j\,\mathbf{t}_j = x_j\,(\mathbf{E}_i \cdot \mathbf{t}_j)\,\mathbf{E}_i \tag{2.5}$$

where the latter equality is obtained after developing $\mathbf{t}_j$ in the basis $\{\mathbf{E}_i\}$. This expression shows that the components of the rotated vector $\mathbf{x}'$ are computed by orthogonal projection on the unit axes:

$$\begin{aligned} x'_1 &= (\mathbf{E}_1 \cdot \mathbf{t}_1)x_1 + (\mathbf{E}_1 \cdot \mathbf{t}_2)x_2 + (\mathbf{E}_1 \cdot \mathbf{t}_3)x_3 \\ x'_2 &= (\mathbf{E}_2 \cdot \mathbf{t}_1)x_1 + (\mathbf{E}_2 \cdot \mathbf{t}_2)x_2 + (\mathbf{E}_2 \cdot \mathbf{t}_3)x_3 \\ x'_3 &= (\mathbf{E}_3 \cdot \mathbf{t}_1)x_1 + (\mathbf{E}_3 \cdot \mathbf{t}_2)x_2 + (\mathbf{E}_3 \cdot \mathbf{t}_3)x_3 \end{aligned} \tag{2.6}$$

giving the expression for the components of the rotation operator in the natural basis $\mathbf{E}_i \otimes \mathbf{E}_j$

$$\mathbf{R} = \begin{bmatrix} (\mathbf{E}_1 \cdot \mathbf{t}_1) & (\mathbf{E}_1 \cdot \mathbf{t}_2) & (\mathbf{E}_1 \cdot \mathbf{t}_3) \\ (\mathbf{E}_2 \cdot \mathbf{t}_1) & (\mathbf{E}_2 \cdot \mathbf{t}_2) & (\mathbf{E}_2 \cdot \mathbf{t}_3) \\ (\mathbf{E}_3 \cdot \mathbf{t}_1) & (\mathbf{E}_3 \cdot \mathbf{t}_2) & (\mathbf{E}_3 \cdot \mathbf{t}_3) \end{bmatrix} \tag{2.7}$$

Let us note that when using this representation

- the dependence of the operator with respect to only 3 parameters does not appear immediately;
- the orthonormality property, on the other hand, is obvious.

Indeed, the inverse transformation

$$\mathbf{x} = \mathbf{R}^{-1}\,\mathbf{x}' \tag{2.8}$$

can also be calculated through the geometric projection

$$\begin{aligned} x_1 &= (\mathbf{t}_1 \cdot \mathbf{E}_1)x'_1 + (\mathbf{t}_1 \cdot \mathbf{E}_2)x'_2 + (\mathbf{t}_1 \cdot \mathbf{E}_3)x'_3 \\ x_2 &= (\mathbf{t}_2 \cdot \mathbf{E}_1)x'_1 + (\mathbf{t}_2 \cdot \mathbf{E}_2)x'_2 + (\mathbf{t}_2 \cdot \mathbf{E}_3)x'_3 \\ x_3 &= (\mathbf{t}_3 \cdot \mathbf{E}_1)x'_1 + (\mathbf{t}_3 \cdot \mathbf{E}_2)x'_2 + (\mathbf{t}_3 \cdot \mathbf{E}_3)x'_3 \end{aligned} \tag{2.9}$$

The scalar product being commutative, one easily verifies that

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

2.3 Finite rotation in terms of dyadic products

A complementary result to (2.4) consists to observe that, using matrix notation, any finite rotation can be written in the form

$$\mathbf{R} = \mathbf{t}_1\,\mathbf{E}_1^T + \mathbf{t}_2\,\mathbf{E}_2^T + \mathbf{t}_3\,\mathbf{E}_3^T = \mathbf{t}_i \otimes \mathbf{E}_i \tag{2.10}$$

This can be shown by simply applying $\mathbf{R}$ to the vector $\mathbf{x}$:

$$\mathbf{R}\,\mathbf{x} = (\mathbf{t}_i \otimes \mathbf{E}_i)\, x_j\, \mathbf{E}_j = x_i\, \mathbf{t}_i = \mathbf{x}' \tag{2.11}$$

Moreover, $\mathbf{R}$ can be defined in this way in terms of any two orthogonal bases $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ and $\{\mathbf{n'}_1, \mathbf{n'}_2, \mathbf{n'}_3\}$, provided these two sets of orthonormal vectors verify the relation:

$$\mathbf{n'}_i = \mathbf{R}\,\mathbf{n}_i \tag{2.12}$$

The proof holds by observing that an arbitrary vector $\mathbf{x}$ is transformed by the so-defined rotation operator according to

$$(\mathbf{n'}_i \otimes \mathbf{n}_i)\,\mathbf{x} = \mathbf{R}\,\mathbf{n}_i\,(\mathbf{n}_i \cdot \mathbf{x}) = \mathbf{R}\,\mathbf{x} = \mathbf{x}' \tag{2.13}$$

Another way to show that the transformation $(\mathbf{n'}_i \otimes \mathbf{n}_i)$ is a rotation is to point out that the length of the vector $\mathbf{x}$ is unaffected by the transformation:

$$x'^2 = ((\mathbf{n'}_j \otimes \mathbf{n}_j)\mathbf{x}) \cdot ((\mathbf{n'}_i \otimes \mathbf{n}_i)\mathbf{x}) = ((\mathbf{n}_j \cdot \mathbf{x})\mathbf{n'}_j) \cdot ((\mathbf{n}_i \cdot \mathbf{x})\mathbf{n'}_i) = (\mathbf{n}_i \cdot \mathbf{x})(\mathbf{n}_i \cdot \mathbf{x}) = x^2 \tag{2.14}$$

2.4 Non-commutative character of finite rotations

Let us consider an object (fig. 5) submitted to two successive rotations $\mathbf{R}_1$ and $\mathbf{R}_2$ of 90° about z and y axes respectively

$$\mathbf{R}_1 = \mathbf{R}(z, 90^\circ) \qquad \text{and} \qquad \mathbf{R}_2 = \mathbf{R}(y, 90^\circ)$$

We know that the matrix product is a non-commutative operation

$$\mathbf{R}_1\,\mathbf{R}_2 \neq \mathbf{R}_2\,\mathbf{R}_1$$

In the finite rotation context, *non-commutativity* expresses the fact that reversing the order of the rotation operations leads to different geometric configurations of the object to which they are applied. It implies that in all decomposition techniques of a finite rotation in terms of elementary rotations as described below, the order in which the successive rotations are performed is essential.

While infinitesimal rotations may be assigned a vectorial entity, this is not the case for finite ones. One important reason is the non-commutative character that hinders the application of the parallelogram composition law, one of the three essential properties of vector quantities. Non-commutativity marks heavily the treatment of finite rotations: we, engineers, are not familiar with this character. Special operation rules are developed

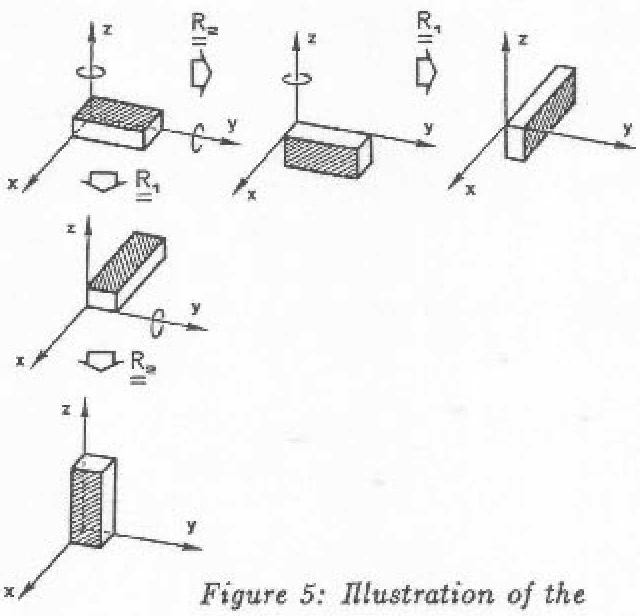

Figure 5: Illustration of the non-commutative character of finite rotations

to handle these objects forming the so-called non-commutative algebras, which will be discussed elsewhere in this report.

2.5 Finite rotations in terms of Euler angles

Euler angles provide a system of 3 independent parameters to express in a unique manner an arbitrary finite rotation. Euler angles are well suited for describing the kinematics of specialized rotating systems such as tops, gyroscopes, etc... It is not so adequate, however, to describe more general articulated systems.

The Euler angle formalism consists of expressing the transformation from $Oxyz$ to $Ox'y'z'$ as a sequence of three elementary rotations (fig. 6)

- a ϕ rotation about Oz: $\mathbf{R}(z, \phi)$
- a θ rotation about Ox_1: $\mathbf{R}(x_1, \theta)$
- a ψ rotation about Oz_2: $\mathbf{R}(z_2, \psi)$

In terms of these 3 successive rotations, the frame transformation can be written

$$\mathbf{x} = \mathbf{R}(z, \phi)\, \mathbf{R}(x_1, \theta)\, \mathbf{R}(z_2, \psi)\, \mathbf{x}' = \mathbf{R}\, \mathbf{x}'$$

with

$$\mathbf{R} = \mathbf{R}(z, \phi)\, \mathbf{R}(x_1, \theta)\, \mathbf{R}(z_2, \psi) \tag{2.15}$$

and where the elementary rotations about z and x axes are given by eqns (2.1) and (2.3).

One obtains explicitly

$$\mathbf{R} = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \tag{2.16}$$

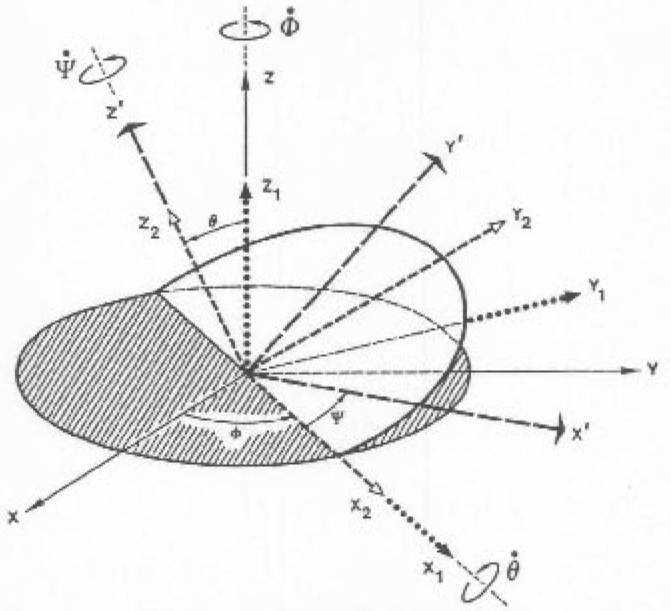

Figure 6: Description of finite rotations in terms of Euler angles

Existence of singularities

The description of finite rotations in terms of Euler angles becomes singular when $\theta = 0$ or π, in which case both intermediate rotation axes along z become collinear: the rotation is then reduced to a single rotation $(\phi \pm \psi)$ about z.

Kinematics inversion

A practical problem that arises frequently in mechanism analysis is the need to solve the *inverse problem*: given the numerical values of $\mathbf{R}$

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \mathbf{R}(\psi, \theta, \phi) \tag{2.17}$$

determine the corresponding numerical values of Euler angles (ψ, θ, ϕ).

Expression (2.16) shows that a crude solution of (2.17) would consist of calculating

$$\theta = \cos^{-1}(r_{33}), \qquad \phi = -\cos^{-1}(\frac{r_{23}}{\sin\theta}), \qquad \psi = \cos^{-1}(\frac{r_{32}}{\sin\theta})$$

This method of solution is however unsatisfactory since it does not provide the sign of the angles and becomes very inaccurate in the vicinity of singular points.

A satisfactory solution consists of making a systematic use of the function $\tan^{-1*}$; one calculates first

$$\psi = \tan^{-1}(\frac{r_{31}}{r_{32}}) \tag{2.18}$$

which gives two possible solutions ψ_1 and $\psi_2 = \psi_1 + \pi$. θ and ϕ may then be evaluated without ambiguity

$$\begin{cases} \sin\theta = r_{31}\sin\psi + r_{32}\cos\psi \\ \cos\theta = r_{33} \end{cases} \tag{2.19}$$

and

$$\begin{cases} \cos\phi = r_{11}\cos\psi - r_{12}\sin\psi \\ \sin\phi = r_{21}\cos\psi - r_{22}\sin\psi \end{cases} \tag{2.20}$$

2.6 Finite rotations in terms of Bryant angles

In order to specify the orientation of some mechanical systems such as a flying vehicle or a Cardan type device, it is better adapted to define the finite rotation operator in terms of three elementary rotations about three distinct axes, called *roll, pitch and yaw* axes.

Let us express the transformation from $Oxyz$ to $Ox'y'z'$ as a sequence of the three elementary rotations (fig. 7)

- a ψ rotation about Oz: $\mathbf{R}(z,\psi)$
- a θ rotation about Oy_1: $\mathbf{R}(y_1,\theta)$
- a ϕ rotation about Ox_2: $\mathbf{R}(x_2,\phi)$

In terms of these 3 successive rotations, the frame transformation can be written

$$\mathbf{x} = \mathbf{R}(z,\psi)\,\mathbf{R}(y_1,\theta)\,\mathbf{R}(x_2,\phi)\mathbf{x}' = \mathbf{R}\,\mathbf{x}'$$

with

$$\mathbf{R} = \mathbf{R}(z,\psi)\,\mathbf{R}(y_1,\theta)\,\mathbf{R}(x_2,\phi) \tag{2.21}$$

where the elementary rotations about z, y_1 and x_2 are given by eqns (2.1), (2.2) and (2.3). One obtains explicitly

$$\mathbf{R} = \begin{bmatrix} \cos\theta\cos\psi & \sin\theta\sin\phi\cos\psi - \sin\psi\cos\phi & \sin\theta\cos\phi\cos\psi + \sin\psi\sin\phi \\ \sin\phi\cos\theta & \sin\theta\sin\phi\sin\psi + \cos\psi\cos\phi & \sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \tag{2.22}$$

* in practice, the FORTRAN function $\psi = ATAN2(x,y)$ provides the result $-\pi \le \psi \le \pi$.

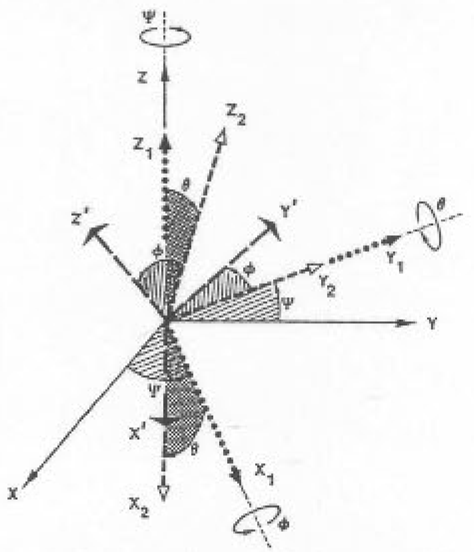

Figure 7: Description of finite rotations in terms of Bryant angles

Existence of singularities

The description of finite rotations in terms of Bryant angles becomes singular when $\theta = \pm\frac{\pi}{2}$, in which case both intermediate rotation axes along z become collinear: the rotation is then reduced to a single rotation $(\phi \pm \psi)$ about z.

Kinematics inversion

The same remark as made for Euler angles holds here. A satisfactory solution consists to calculate first

$$\psi = \tan^{-1}(\frac{r_{21}}{r_{11}}) \tag{2.23}$$

which gives two possible solutions ψ_1 and $\psi_2 = \psi_1 + \pi$. θ and ϕ may then be evaluated without ambiguity

$$\begin{cases} \cos\theta = r_{21}\sin\psi + r_{11}\cos\psi \\ \sin\theta = -r_{31} \end{cases} \tag{2.24}$$

and

$$\begin{cases} \cos\phi = (r_{13}\cos\psi + r_{23}\sin\psi)\sin\theta + r_{33}\cos\theta \\ \sin\phi = (r_{12}\cos\psi + r_{22}\sin\psi)\sin\theta + r_{32}\cos\theta \end{cases} \tag{2.25}$$

2.7 Finite rotation as a unique rotation about an arbitrary axis

Both types of rotation parametrizations seen in paragraphs 2.5 and 2.6 are clear examples of non–invariant measures. They were developed to treat particular cases, i.e. the Euler angles are oriented to describe the kinematics of the spinning top problem, and they are not well suited to handle any general problem. In this paragraph and in the next one, we will describe the geometrical derivation of two *invariant parametrizations*, that is, parametrizations that do not possess any preferred direction.

Euler's theorem on finite rotations (see paragraph 1.2) states that any finite rotation can be expressed as a unique rotation ϕ about an appropriate axis $\vec{n}$ (fig. 8).

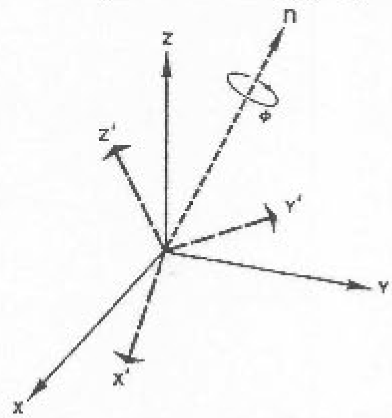

Figure 8: Expression of a finite rotation as a unique rotation ϕ about axis $\vec{n}$

In this case, there are 4 rotation parameters, which are however linked by one constraint:

$$n_x,\ n_y,\ n_z,\ \phi \tag{2.26}$$

with

$$\|\mathbf{n}\| = \sqrt{n_x^2 + n_y^2 + n_z^2} = 1, \qquad \phi \in [0, \pi]$$

The most intuitive procedure to obtain the explicit expression of the corresponding rotation operator consists to proceed in 3 phases

(a) 2 elementary successive rotations (fig. 9)

$$\mathbf{R}(z, -\alpha) \qquad \text{and} \qquad \mathbf{R}(y, +\beta)$$

combined in a unique rotation matrix

$$\mathbf{C} = \mathbf{R}(z, -\alpha)\, \mathbf{R}(y, +\beta) \tag{2.27}$$

have the effect of bringing the $\vec{n}$ axis in coincidence with the Ox axis

(b) perform rotation ϕ about the Ox axis

(c) bring the $\vec{n}$ axis into position by the inverse transformation $\mathbf{C}^{-1} = \mathbf{C}^T$

The resulting operation combines into

$$\mathbf{R}(\vec{n}, \phi) = \mathbf{C}\, \mathbf{R}(x, \phi)\, \mathbf{C}^T \tag{2.28}$$

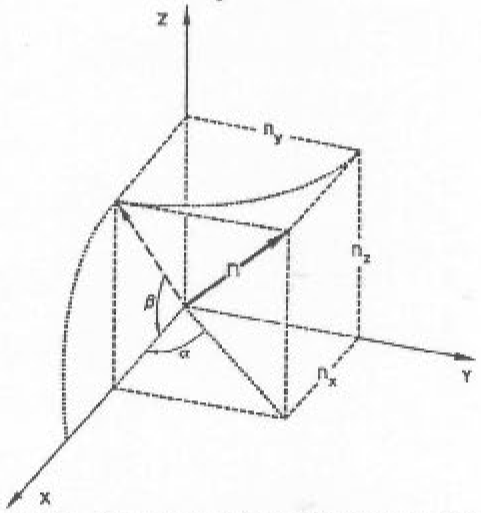

Figure 9: Superposition of two vectors through two successive rotations

with the matrix

$$\mathbf{C} = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha & \cos\alpha\sin\beta \\ -\sin\alpha\cos\beta & \cos\alpha & -\sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.29}$$

If one notes further that

$$\begin{aligned} \sin\beta &= \sqrt{n_x^2 + n_y^2} & \cos\beta &= \sqrt{n_x^2 + n_y^2} \\ \sin\alpha &= \frac{n_y}{\sqrt{n_x^2 + n_y^2}} & \cos\alpha &= \frac{n_x}{\sqrt{n_x^2 + n_y^2}} \end{aligned} \tag{2.30}$$

one obtains the explicit expression for the rotation operator

$$\mathbf{R} = \begin{bmatrix} n_x^2\, V\phi + C\phi & n_x n_y\, V\phi - n_z\, S\phi & n_x n_z\, V\phi + n_y\, S\phi \\ n_x n_y\, V\phi + n_z\, S\phi & n_y^2\, V\phi + C\phi & n_y n_z\, V\phi - n_x\, S\phi \\ n_x n_z\, V\phi - n_y\, S\phi & n_y n_z\, V\phi + n_x\, S\phi & n_z^2\, V\phi + C\phi \end{bmatrix} \tag{2.31}$$

with the notations

$$C\phi = \cos\phi, \qquad S\phi = \sin\phi, \qquad V\phi = \mathrm{vers}(\phi) = 1 - \cos\phi \tag{2.32}$$

An alternate proof of this result can be obtained by starting from eqn (2.13), from which we express the rotation operator in the form:

$$\mathbf{R} = [\mathbf{n'}_1\, \mathbf{n}_1^T + \mathbf{n'}_2\, \mathbf{n}_2^T + \mathbf{n'}_3\, \mathbf{n}_3^T] \tag{2.33}$$

Suppose that $\mathbf{n}_1$ is oriented along the axis of rotation. Then, vectors $\mathbf{n}_2$ and $\mathbf{n}_3$ are transformed according to

$$\begin{aligned} \mathbf{n'}_2 &= \mathbf{n}_2\cos\phi + \mathbf{n}_3\sin\phi \\ \mathbf{n'}_3 &= -\mathbf{n}_2\sin\phi + \mathbf{n}_3\cos\phi \end{aligned} \tag{2.34}$$

and substitution of (2.34) into (2.33) yields to

$$\mathbf{R} = [\mathbf{n'}_1\, \mathbf{n}_1^T + \cos\phi\ (\mathbf{n}_2\, \mathbf{n}_2^T + \mathbf{n}_3\, \mathbf{n}_3^T) + \sin\phi\ (\mathbf{n}_2\, \mathbf{n}_3^T - \mathbf{n}_3\, \mathbf{n}_2^T)] \tag{2.35}$$

If we further note that $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ form an orthonormal basis, the following identity can be established:

$$\mathbf{n}_1 \times \mathbf{h} = (\mathbf{n}_2 \times \mathbf{n}_3) \times \mathbf{h} = (\mathbf{n}_3 \cdot \mathbf{h})\, \mathbf{n}_2 - (\mathbf{n}_2 \cdot \mathbf{h})\, \mathbf{n}_3 = (\mathbf{n}_2 \otimes \mathbf{n}_3 - \mathbf{n}_3 \otimes \mathbf{n}_2)\, \mathbf{h} \qquad \forall \mathbf{h} \in \Re^3 \tag{2.36}$$

Then,

$$(\mathbf{n}_2\, \mathbf{n}_3^T - \mathbf{n}_3\, \mathbf{n}_2^T) = \tilde{\mathbf{n}}_1 \tag{2.37}$$

where $\tilde{\mathbf{n}}_1$ is the skew-symmetric matrix

$$\tilde{\mathbf{n}}_1 = \begin{bmatrix} 0 & -n_{1\,z} & n_{1\,y} \\ n_{1\,z} & 0 & -n_{1\,x} \\ -n_{1\,y} & n_{1\,x} & 0 \end{bmatrix} \tag{2.38}$$

We can also easily verify the identity:

$$\mathbf{n}_3 \otimes \mathbf{n}_3 + \mathbf{n}_2 \otimes \mathbf{n}_2 = \mathbf{1} - \mathbf{n}_1 \otimes \mathbf{n}_1 \tag{2.39}$$

The resulting rotation operator is then equivalent to eqn (2.31), written in matrix form

$$\mathbf{R} = [\cos\phi\ \mathbf{1} + (1 - \cos\phi)\ \mathbf{n}\ \mathbf{n}^T + \sin\phi\ \tilde{\mathbf{n}}] \tag{2.40}$$

where the index 1 was eliminated for conciseness.

Kinematics inversion

When trying to invert (2.31) in closed form, one faces again the problem of an existing singularity. The sum of the diagonal terms in eqn (2.31), when compared to (2.11), provides the relationship

$$\begin{aligned} r_{11} + r_{22} + r_{33} &= (n_x^2 + n_y^2 + n_z^2)\ (1 - \cos\phi) + 3\cos\phi \\ &= 1 + 2\cos\phi \end{aligned} \tag{2.41}$$

and similarly, the quantity obtained from the differences of off-diagonal terms gives

$$\begin{aligned} (r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2 &= 4(n_x^2 + n_y^2 + n_z^2)\ \sin^2\phi \\ &= 4\sin^2\phi \end{aligned} \tag{2.42}$$

so that the rotation angle ϕ can be calculated by

$$\phi = \tan^{-1} \frac{\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2}}{r_{11} + r_{22} + r_{33} - 1} \qquad \phi \in [0, \pi] \tag{2.43}$$

and the difference of the off diagonal terms gives next

$$n_x = \frac{(r_{32} - r_{23})}{2\sin\phi}, \qquad n_y = \frac{(r_{13} - r_{31})}{2\sin\phi}, \qquad n_z = \frac{(r_{21} - r_{12})}{2\sin\phi} \tag{2.44}$$

The relations (2.43-44) show that for a rotation angle close to zero, the determination of the rotation direction becomes very inaccurate. This difficulty can be overcome only by not distinguishing between the rotation direction and the corresponding angle, as Euler parameters do. They are described in the next section.

2.8 Finite rotations in terms of Euler parameters

Let us introduce the following definitions

$$e_0 = \cos\frac{\phi}{2} \qquad \begin{aligned} e_1 &= n_x \sin\frac{\phi}{2} \\ e_2 &= n_y \sin\frac{\phi}{2} \\ e_3 &= n_z \sin\frac{\phi}{2} \end{aligned} \tag{2.45}$$

Equation (2.31) may then be rewritten in the form

$$\mathbf{R} = \begin{bmatrix} 1 - 2({e_2}^2 + {e_3}^2) & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & 1 - 2({e_1}^2 + {e_3}^2) & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & 1 - 2({e_1}^2 + {e_2}^2) \end{bmatrix} \tag{2.46}$$

where the four parameters introduced are algebraic quantities which play equal roles. they are linked by the constraint

$${e_0}^2 + {e_1}^2 + {e_2}^2 + {e_3}^2 = 1 \tag{2.47}$$

Even though the component e_0 is a known function of the other three components, the four components are needed for accurate computation of the matrix $\mathbf{R}$ (when ϕ is nearly 180 deg., the magnitude of $\mathbf{e}$, i.e. $\sin\phi/2$ becomes insensitive to variations in ϕ).

The kinematic inversion does not give rise to any singularity (in fact there is a two-to-one relation between Euler parameters and rotations, because $(e_0, \mathbf{e})$ and $(-e_0, -\mathbf{e})$ represent the same rotation; so, the sign of Euler parameters has to be arbitrarily chosen when they are derived from $\mathbf{R}$ and also, the positive and negative values have to be tested for equality of rotations). An algebraic manipulation of the components of (2.46) together with the constraint provides the inversion formulas

$$\begin{aligned} e_0 &= \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} \\ e_1 &= \frac{1}{2}\mathrm{sign}(r_{32} - r_{23})\ \sqrt{1 + r_{11} - r_{22} - r_{33}} \\ e_2 &= \frac{1}{2}\mathrm{sign}(r_{13} - r_{31})\ \sqrt{1 - r_{11} + r_{22} - r_{33}} \\ e_3 &= \frac{1}{2}\mathrm{sign}(r_{21} - r_{12})\ \sqrt{1 - r_{11} - r_{22} + r_{33}} \end{aligned} \tag{2.48}$$

One observes thus that Euler parameters form a set of four dependent parameters, and their dependence property is their main drawback. They have however some very nice properties which make their use attractive:

- the associated inversion procedure does not give rise to any singularity;
- they are algebraic quantities: the use of transcendental functions appears only when separating the rotation angle from the rotation direction is necessary;
- in calculations where Euler parameters are used, their property of obeying to the so-called quaternion multiplication rule (see section 4) brings significant simplifications to the arithmetic operations.

3. MATRIX APPROACH TO FINITE ROTATIONS

3.1 Cayley form of an orthogonal matrix

Let us start again from the fact that the finite rotation of a vector about the origin may be described by the operation

$$\mathbf{x}' = \mathbf{R}\,\mathbf{x} \tag{3.1}$$

where $\mathbf{R}$ is an orthogonal matrix. This operation conserves the length of the original vector,

$$\mathbf{x}'^T\mathbf{x}' - \mathbf{x}^T\mathbf{x} = 0 \tag{3.2}$$

or

$$(\mathbf{x}' + \mathbf{x})^T(\mathbf{x}' - \mathbf{x}) = 0 \tag{3.3}$$

Equation (3.3) means that the vectors $\mathbf{f}$ and $\mathbf{g}$ defined by

$$\begin{aligned} \mathbf{f} &= \mathbf{x}' - \mathbf{x} = (\mathbf{R} - \mathbf{1})\,\mathbf{x} \\ \mathbf{g} &= \mathbf{x}' + \mathbf{x} = (\mathbf{R} + \mathbf{1})\,\mathbf{x} \end{aligned} \tag{3.4}$$

are orthogonal together

$$\mathbf{f}^T\,\mathbf{g} = 0 \tag{3.5}$$

Let us next eliminate $\mathbf{x}$ between both equations (3.4). One obtains

$$\mathbf{f} = (\mathbf{R} - \mathbf{1})\,(\mathbf{R} + \mathbf{1})^{-1}\,\mathbf{g} = \mathbf{B}\,\mathbf{g} \tag{3.6}$$

where $\mathbf{B}$ is necessarily of antisymmetric type, since (3.5) yields to

$$\mathbf{g}^T\mathbf{B}\,\mathbf{g} = 0 \tag{3.7}$$

Let us express this property in terms of the vector $\mathbf{b}^T = \langle\, b_1 \quad b_2 \quad b_3 \,\rangle$ collecting the components of the matrix

$$\mathbf{B} \;=\; \tilde{\mathbf{b}} \;=\; \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \tag{3.8}$$

Equation (3.6) shows that the rotation matrix $\mathbf{R}$ is such that

$$\mathbf{R} \,-\, \mathbf{1} \;=\; \tilde{\mathbf{b}}\,(\mathbf{R} \,+\, \mathbf{1}) \tag{3.9}$$

If we solve it with respect to $\mathbf{R}$, one gets the relationship

$$\mathbf{R} \;=\; (\mathbf{1} \,-\, \tilde{\mathbf{b}})^{-1}\,(\mathbf{1} \,+\, \tilde{\mathbf{b}}) \tag{3.10}$$

which corresponds to a particular choice of the three parameters describing the finite rotation.

The following identity, which can be easily verified by simple computation, will be repeatedly used in this report. Let $\mathbf{h} \in \Re^3$; then:

$$(\alpha\,\mathbf{1} \,+\, \beta\,\mathbf{h}\otimes\mathbf{h} \,+\, \gamma\,\tilde{\mathbf{h}})^{-1} \;=\; (\alpha_1\,\mathbf{1} \,+\, \beta_1\,\mathbf{h}\otimes\mathbf{h} \,+\, \gamma_1\,\tilde{\mathbf{h}}) \tag{3.11}$$

with

$$\begin{aligned} \alpha_1 &= \frac{\alpha}{\gamma^2\,\|\mathbf{h}\|^2 \,+\, \alpha^2} \\ \beta_1 &= \frac{(\gamma^2 \,-\, \alpha\,\beta)\,\|\mathbf{h}\|^2}{(\gamma^2\,\|\mathbf{h}\|^2 \,+\, \alpha^2)\,(\alpha \,+\, \beta\,\|\mathbf{h}\|^2)} \\ \gamma_1 &= \frac{-\gamma}{\gamma^2\,\|\mathbf{h}\|^2 \,+\, \alpha^2} \end{aligned} \tag{3.12}$$

Let us then calculate explicitly

$$(\mathbf{1} \,-\, \tilde{\mathbf{b}})^{-1} \;=\; \frac{1}{1 \,+\, \|\mathbf{b}\|^2}\,[\mathbf{1} \,+\, \mathbf{b}\otimes\mathbf{b} \,+\, \tilde{\mathbf{b}}] \tag{3.13}$$

Performing the product (3.10) leads to the general algebraic expression for the rotation operator

$$\mathbf{R} \;=\; \frac{1}{1 \,+\, \|\mathbf{b}\|^2}\,[(1 - \|\mathbf{b}\|^2)\,\mathbf{1} \,+\, 2\,\mathbf{b}\otimes\mathbf{b} \,+\, 2\,\tilde{\mathbf{b}}] \tag{3.14}$$

By making use of the fact that

$$\mathbf{b}\otimes\mathbf{b} \,-\, \|\mathbf{b}\|^2\,\mathbf{1} \;=\; \tilde{\mathbf{b}}\,\tilde{\mathbf{b}} \tag{3.15}$$

equation (3.14) may be rewritten in the more compact form

$$\mathbf{R} \;=\; \mathbf{1} \,+\, \frac{2}{1 \,+\, \|\mathbf{b}\|^2}\,(\tilde{\mathbf{b}} \,+\, \tilde{\mathbf{b}}\,\tilde{\mathbf{b}}) \tag{3.16}$$

3.2 Possible choices of the parameters

In the following, we discuss several alternatives of invariant measures proposed in the literature to parametrize the rotation operator. They can all be related, in a more or less direct way, to equations (3.14-16). We qualify them as invariant, to differentiate from parameters with preferential directions, i.e. Euler angles and Bryant angles.

- *Rodrigues parameters*

The set of three parameters b_i are usually called Rodrigues parameters. Their geometrical meaning can be obtained by comparing (3.14) to (2.40). By equating coefficients corresponding to the term $\tilde{\mathbf{n}}$

$$\sin(\phi) = \frac{2\,\|\mathbf{b}\|}{1 + \|\mathbf{b}\|^2} \tag{3.17}$$

and by performing some algebraic steps, we arrive at the result:

$$\|\mathbf{b}\| = \tan\frac{\phi}{2} \tag{3.18}$$

from which we obtain

$$\begin{cases} b_1 = n_x \tan\frac{\phi}{2} \\ b_2 = n_y \tan\frac{\phi}{2} \\ b_3 = n_z \tan\frac{\phi}{2} \end{cases} \tag{3.19}$$

The Rodrigues parameters offer the advantage of using just three independent quantities. However, they give rise to a singularity when $\phi = \pm\pi$ and have thus to be used with caution.

- *Euler parameters*

By making the change of variables

$$b_i = \frac{e_i}{e_0} \qquad i = 1, 2, 3 \tag{3.20}$$

and by defining the fourth parameter e_0 through the normality condition

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \tag{3.21}$$

we restore the rotation operator (2.46) obtained from geometric considerations

$$\begin{aligned} \mathbf{R} &= (2e_0^2 - 1)\,\mathbf{1} + 2\,(\mathbf{e}\,\mathbf{e}^T + e_0\,\tilde{\mathbf{e}}) \\ &= \mathbf{1} + 2\,e_0\,\tilde{\mathbf{e}} + 2\,\tilde{\mathbf{e}}^2 \end{aligned} \tag{3.22}$$

The Euler parameters have the geometric meaning

$$e_0 = \cos\frac{\phi}{2} \qquad \begin{cases} e_1 = n_x \sin\frac{\phi}{2} \\ e_2 = n_y \sin\frac{\phi}{2} \\ e_3 = n_z \sin\frac{\phi}{2} \end{cases} \tag{3.23}$$

in terms of the rotation angle $\phi \in [-\pi, +\pi]$ about the direction $\mathbf{n}$ (this fact can be verified easily from (3.19) and (3.20-21)). They are such that

$$-1 \leq e_i \leq 1 \qquad i = 0, 1, 2, 3. \tag{3.24}$$

- *The conformal rotation vector (CRV)*

The idea of the conformal rotation vector seems to have been introduced for the first time by Milenkovic [2], and developed further in ref.[3]. It is based on a conformal transformation on Euler parameters

$$c_i \;=\; \frac{4e_i}{1 \;+\; e_0} \qquad i = 0, 1, 2, 3 \tag{3.25}$$

which produces a set of three independent parameters

$$\begin{cases} c_1 \;=\; \frac{4e_0 b_1}{1+e_0} \;=\; 4\; n_x \tan\frac{\phi}{4} \\ c_2 \;=\; \frac{4e_0 b_2}{1+e_0} \;=\; 4\; n_y \tan\frac{\phi}{4} \\ c_3 \;=\; \frac{4e_0 b_3}{1+e_0} \;=\; 4\; n_z \tan\frac{\phi}{4} \end{cases} \qquad \phi \in [-\pi, +\pi] \tag{3.26}$$

The fourth parameter is given by

$$c_0 \;=\; \frac{1}{8}[16 \;-\mathbf{c}^T\mathbf{c}] \;=\; \frac{1}{8}[16 \;-\; \|\mathbf{c}\|^2] \tag{3.27}$$

Compared to the Rodrigues parameters, they do not contain any singularity in the interval $\phi \in [-\pi, +\pi]$, since they are such that

$$\begin{aligned} 0 \leq & c_0 \leq 2 \\ -4 \leq & c_i \leq +4 \qquad i = 1, 2, 3 \end{aligned} \tag{3.28}$$

The formulas (3.25) can still be inverted in the form

$$e_i \;=\; \frac{c_i}{4 - c_0} \qquad i = 0, 1, 2, 3 \tag{3.29}$$

and provide the expression of the rotation matrix

$$\mathbf{R} \;=\; \frac{1}{(4 - c_0)^2}\,[\;2\;\mathbf{c}\;\mathbf{c}^T \;+\; 2\;c_0\;\tilde{\mathbf{c}} \;+\; (c_0^2 - \|\mathbf{c}\|^2)\;\mathbf{1}] \tag{3.30}$$

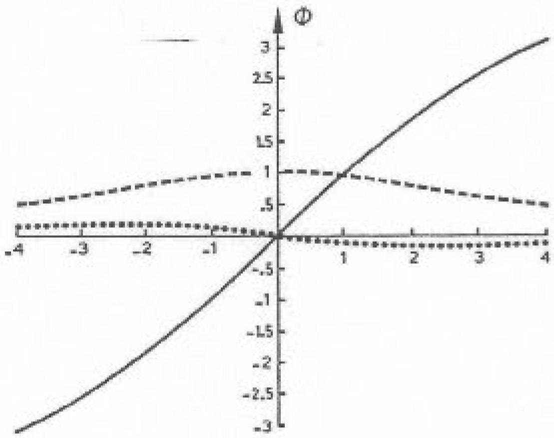

Figure 10: Variation of rotation angle and its derivatives in terms of CRV parameters

or, if we use $\tilde{c}\tilde{c} = \mathbf{c}\mathbf{c}^T - \|\mathbf{c}\|^2\mathbf{1}$, we obtain

$$\mathbf{R} = \frac{1}{(4-c_0)^2}\,[\,2\,c_0\,\tilde{c} + \mathbf{c}\,\mathbf{c}^T + c_0^2\,\mathbf{1} + \tilde{c}\,\tilde{c}]$$

Finally, if we make also use of the facts

$$16 = 8c_0 + \|\mathbf{c}\|^2 \qquad \text{and} \qquad \tilde{c}\,\mathbf{c} = 0 \tag{3.31}$$

we get the expression

$$\mathbf{R} = \frac{1}{(4-c_0)^2}\,[c_0^2\,\mathbf{1} + \frac{1}{2}\,c_0\,\mathbf{c}\,\mathbf{c}^T + \frac{1}{16}\,\|\mathbf{c}\|^2\,\mathbf{c}\,\mathbf{c}^T + 2\,c_0\,\tilde{c} + \tilde{c}\,\tilde{c}] \tag{3.32}$$

The latter equation can be rewritten in the compact form

$$\mathbf{R} = \mathbf{F}^2 \tag{3.33}$$

with the operator

$$\mathbf{F} = \frac{1}{(4-c_0)}\,[c_0\,\mathbf{1} + \frac{1}{4}\,\mathbf{c}\,\mathbf{c}^T + \tilde{c}] \tag{3.34}$$

which has the meaning of a rotation of $\phi/2$ expressed in terms of the Rodrigues parameters of value $b_i = c_i/4$. In this way, the total rotation is regarded as the composition of two equal partial rotations.

The motivation for introducing the factor 4 into the expression of CRV's, appears clearly when taking the variation of (3.26). In this way, the variations of CRV parameters have the meaning of infinitesimal angles

$$\delta\mathbf{c} = \mathbf{n}\,\delta\phi = \delta\boldsymbol{\phi} \tag{3.35}$$

An important feature of CRV parameters is the almost linearity of the relationship $\phi(c)$ in the interval $[-\pi, +\pi]$. It is illustrated by fig. 10 which displays the angle and its first and second derivatives in terms of c.

- *The rotational vector*

Let us introduce in (3.16) the explicit expression (3.19) of Rodrigues parameters. The rotation operator takes then the form

$$\mathbf{R} = \mathbf{1} + \frac{2}{1 + \tan^2 \frac{\phi}{2}} \left(\tan \frac{\phi}{2} \, \tilde{\mathrm{n}} + \tan^2 \frac{\phi}{2} \, \tilde{\mathrm{n}} \, \tilde{\mathrm{n}} \right) \tag{3.36}$$

Let us next define the *rotational vector* by the vector quantity

$$\boldsymbol{\phi} = \mathbf{n} \, \phi \tag{3.37}$$

made of the rotation components ϕ_x, ϕ_y and ϕ_z about the rotation direction n. Starting from (3.36), the rotation operator may be written in the form

$$\mathbf{R} = \mathbf{1} + \frac{\sin \phi}{\phi} \, \tilde{\phi} + \frac{1}{2} \left(\frac{\sin \frac{\phi}{2}}{\frac{\phi}{2}} \right)^2 \tilde{\phi} \, \tilde{\phi} \tag{3.38}$$

The following alternative expression (see 2.40) can be obtained after some algebraic steps

$$\mathbf{R} = \cos \phi \, \mathbf{1} + \frac{1 - \cos \phi}{\phi^2} \, \phi \, \phi^T + \frac{\sin \phi}{\phi} \, \tilde{\phi} \tag{3.39}$$

A highly simplified expression can be computed from (3.38), by noting that the factors $\frac{\sin \phi}{\phi}$ and $\frac{1}{2}(\frac{\sin \frac{\phi}{2}}{\frac{\phi}{2}})^2$ can be expanded in power series, giving [4]

$$\begin{aligned} \mathbf{R} = \mathbf{1} &+ \left(1 - \frac{\phi^2}{3!} + \frac{\phi^4}{5!} \ldots + (-1)^n \frac{\phi^{2n}}{2n+1!} \pm \ldots \right) \tilde{\phi} \\ &+ \left(\frac{1}{2!} - \frac{\phi^2}{4!} + \frac{\phi^4}{6!} \ldots + (-1)^n \frac{\phi^{2n}}{2n+2!} \pm \ldots \right) \tilde{\phi}\tilde{\phi} \end{aligned} \tag{3.40}$$

Next we observe the powers of $\tilde{\phi}$ and confirm by simple matrix multiplications the interesting relations

$$\begin{aligned} \tilde{\phi}^3 &= (\phi\phi^T - \phi^2 \, \mathbf{1}) \, \tilde{\phi} = -\phi^2 \, \tilde{\phi} \\ \tilde{\phi}^4 &= \tilde{\phi}^3 \tilde{\phi} = -\phi^2 \, \tilde{\phi}^2 \\ \tilde{\phi}^5 &= \phi^4 \, \tilde{\phi} \\ \tilde{\phi}^6 &= \phi^4 \, \tilde{\phi}^2 \end{aligned} \tag{3.41}$$

leading to the recurrence formulas

$$\begin{aligned} \tilde{\phi}^{2n-1} &= (-1)^{2n-1} \phi^{2(n-1)} \, \tilde{\phi} \\ \tilde{\phi}^{2n} &= (-1)^{n-1} \phi^{2(n-1)} \, \tilde{\phi}^2 \end{aligned} \tag{3.42}$$

Applying (3.41) to (3.39) we deduce immediately the series representation of $\mathbf{R}$

$$\mathbf{R} = \mathbf{1} + \tilde{\phi} + \frac{1}{2!} \tilde{\phi}^2 + \frac{1}{3!} \tilde{\phi}^3 + \ldots + \frac{1}{n!} \tilde{\phi}^n \ldots \tag{3.43}$$

which is, in fact, the exponential expansion

$$\mathbf{R} = \exp\tilde{\phi} = e^{\tilde{\phi}} \tag{3.44}$$

In the following, we will refer to this expression as the *exponential map* of rotations.

A direct proof of (3.43) can also be obtained in the following way. Suppose that we decompose $\mathbf{R}$ into a sequence of infinitesimal rotations $\mathbf{\Delta}$ of equal amplitude about the same vector $\mathbf{n}$

$$\mathbf{\Delta} = \lim_{n\to\infty}[\mathbf{1} + \frac{\tilde{\phi}}{n}] \tag{3.45}$$

The rotation operator can then be expressed in the form

$$\mathbf{R} = \lim_{n\to\infty}\mathbf{\Delta}^n = \lim_{n\to\infty}[\mathbf{1} + \frac{\tilde{\phi}}{n}]^n \tag{3.46}$$

By developing this expression using the binomial theorem and taking the limit we confirm immediately the series (3.43) and hence (3.44).

- *Linear parameters*

Another parametrization presented in the literature are the so-called *linear parameters*. They can be deduced starting from the expression (3.38) for the rotation operator; this equation can be rewritten in the form:

$$\mathbf{R} = \cos\phi\,\mathbf{1} + \frac{\sin^2\phi}{1+\cos\phi}\,\frac{\phi\,\phi^T}{\phi^2} + \sin\phi\,\frac{\tilde{\phi}}{\phi}$$

Then by defining

$$s_0 = \cos\phi \qquad \begin{cases} s_1 = n_x\ \sin\phi \\ s_2 = n_y\ \sin\phi \\ s_3 = n_z\ \sin\phi \end{cases} \tag{3.47}$$

the rotation operator is written

$$\mathbf{R} = s_0\,\mathbf{1} + \frac{1}{1+s_0}\,\mathbf{s}\,\mathbf{s}^T + \tilde{\mathbf{s}} \tag{3.48}$$

The linear parameters are mutually related through the constraint:

$$s_0^2 + s_1^2 + s_2^2 + s_3^2 = 1 \tag{3.49}$$

They constitute a set of four parameters as Euler's, but they have the disadvantage of presenting a singularity point at $\phi = \pm\pi$. Also, the composition rule is rather involved since it can not be related to the quaternion product (see paragraph 4.2).

Data: EULER

e_0, e_1, e_2, e_3

$$E_i = 2e_i \quad (i = 1,2,3)$$

$$\mathbf{R} = \begin{bmatrix} 1 - e_2E_2 - e_3E_3 & e_1E_2 - e_0E_3 & e_1E_3 + e_0E_2 \\ e_1E_2 + e_0E_3 & 1 - e_1E_1 - e_3E_3 & e_2E_3 - e_0E_1 \\ e_1E_3 - e_0E_2 & e_2E_3 + e_0E_1 & 1 - e_1E_1 - e_2E_2 \end{bmatrix}$$

Figure 11: Numerical evaluation of the rotation matrix from Euler parameters

3.3 Numerical evaluation of the rotation matrix

Of all the parametrization systems discussed in the latter paragraph, Rodrigues and linear parameters are of limited applicability in practice due to the singularity that they present for angles $\phi = \pm\pi$. We will analyze next the practical computation of the rotation operator in terms of the other three sets of parameters.

Given the Euler parameters, the flowchart of fig. 11 describes the computational procedure to evaluate the rotation matrix $\mathbf{R}$, where use is made of the Euler parameter expression (3.22). Starting from Euler parameters, 12 multiplies and 12 adds have to be performed.

Similar flowcharts can be formed for the conformal rotation vector and for the rotational vector, now using eqns (3.30) and (3.39) respectively. Starting from the conformal rotation vector, 17 multiplies and 13 adds are required. When computing the rotation operator from the rotational vector, the cost is higher because some trigonometric functions should be evaluated. The total number of computations includes 2 trigonometric evaluations, 1 square root, 18 multiplies and 12 adds.

3.4 Evaluation of the rotation parameters from the R matrix

It is easy to verify that the 4×4 symmetric matrix $\mathbf{S}$ obtained from the individual terms of $\mathbf{R}$

$$\mathbf{S} = \begin{bmatrix} 1 + r_{11} + r_{22} + r_{33} & r_{32} - r_{23} & r_{13} - r_{31} & r_{21} - r_{12} \\ r_{32} - r_{23} & 1 + r_{11} - r_{22} - r_{33} & r_{12} + r_{21} & r_{13} + r_{31} \\ r_{13} - r_{31} & r_{21} + r_{12} & 1 - r_{11} + r_{22} - r_{33} & r_{23} + r_{32} \\ r_{21} - r_{12} & r_{13} + r_{31} & r_{23} + r_{32} & 1 - r_{11} - r_{22} + r_{33} \end{bmatrix} \tag{3.50}$$

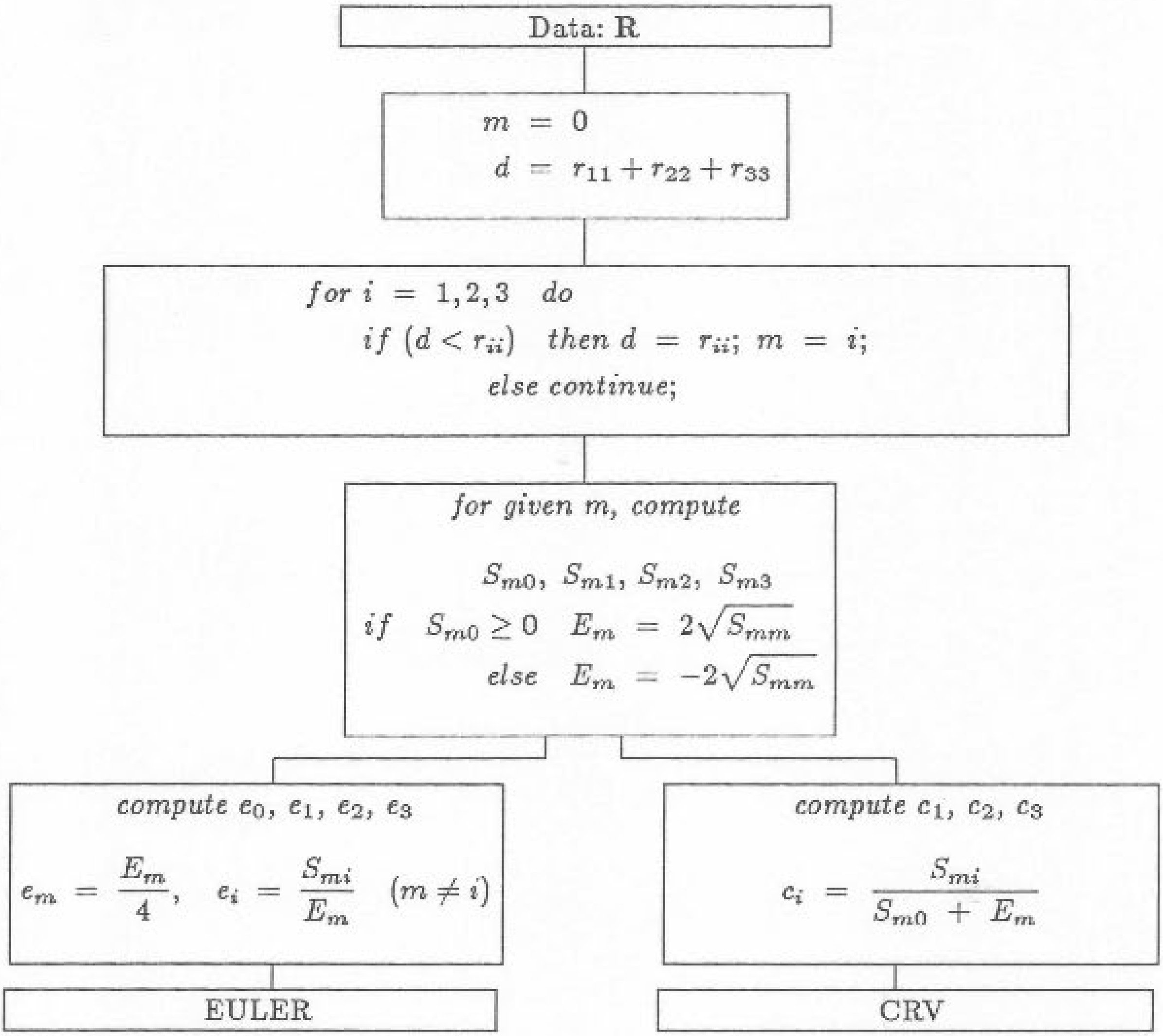

Figure 12: Numerical evaluation of the rotation parameters from the rotation matrix $\mathbf{R}$

is a quadratic expression in terms of Euler Parameters

$$\mathbf{S} = 4 \begin{bmatrix} e_0^2 & e_0e_1 & e_0e_2 & e_0e_3 \\ e_0e_1 & e_1^2 & e_1e_2 & e_1e_3 \\ e_0e_2 & e_1e_2 & e_2^2 & e_2e_3 \\ e_0e_3 & e_1e_3 & e_2e_3 & e_3^2 \end{bmatrix} \tag{3.51}$$

The knowledge of one single row of $\mathbf{S}$ allows thus to compute the rotation parameters.

A possible computational procedure which guarantees at the same time the uniqueness of the solution and a maximum accuracy is given by Spurrier's [5] algorithm, summarized on the flowchart of fig. 12 (we note that in this figure, the rows and columns of $\mathbf{S}$ are indexed as $0, 1, 2, 3$). This procedure consists of using the row with the largest diagonal entry in $\mathbf{S}$. It involves 4 multiplies, 9 adds and a square root extraction to evaluate the Euler parameters.

The rotational vector is evaluated from the computed values of the CRV parameters:

$$\phi = \frac{4}{\|\mathbf{c}\|} \tan^{-1}\left(\frac{\|\mathbf{c}\|}{4}\right) \mathbf{c} \tag{3.52}$$

The evaluation is made from the CRV's and not from the Euler parameters in order to obtain maximum accuracy in the computation. Since the sine and cosine functions are insensitive to angle variations for values near $\pi/2$ and 0, respectively, the evaluation of ϕ in terms of Euler parameters may introduce numerical errors.

3.5 Scalar and vector parts of the rotation matrix

It is interesting to note that by examining the first column of matrix S, one arrives directly to the concept of separation of the rotation operator in its vector part and its scalar part. Let $\mathbf{q}$ and q_0 be

$$\begin{aligned} \mathbf{q} &= \mathrm{Vect}(\mathbf{R}) \\ q_0 &= \mathrm{tr}(\mathbf{R}) \end{aligned} \tag{3.53}$$

such that

$$\begin{aligned} q_i &= \frac{1}{2} \epsilon_{ijk}\, r_{kj} \\ q_0 &= r_{11} + r_{22} + r_{33} \end{aligned} \tag{3.54}$$

Their geometrical interpretation can be obtained from (2.31):

$$\begin{aligned} \mathbf{q} &= \mathbf{n} \sin\phi \\ q_0 &= 1 + 2\cos\phi \end{aligned} \tag{3.55}$$

They are directly related to the linear parameters:

$$\begin{aligned} \mathbf{s} &= \mathbf{q} \\ c &= \frac{1}{2}(q_0 - 1) \end{aligned} \tag{3.56}$$

Euler parameters can be deduced from the vector an scalar parts of the rotation:

$$\begin{aligned} e_0 &= \pm\sqrt{\frac{1+q_0}{2}} \\ e_i &= \frac{q_i}{2e_0} \end{aligned} \tag{3.57}$$

This formulas correspond to a particular case of the procedure previously described. However, they could lead to a bad conditioning of computations, since there is no choice of a maximal pivot.

4. THE ALGEBRAIC APPROACH: QUATERNION ALGEBRA

4.1 Elements of quaternion algebra [6,7]

The algebra of quaternions was introduced by Hamilton [8]. It has only recently been carried to practice in industry, i.e. to model robotics problems and different aerospace applications. A quaternion is defined as a 4−dimensional complex number

$$\widehat{q} = q_0 + iq_1 + jq_2 + kq_3 \tag{4.1}$$

with i, j and k being imaginary unit numbers such that

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ jk &= -kj = i \\ ki &= -ik = j \\ ij &= -ji = k \end{aligned}$$

It can be alternatively written in vector notation

$$\widehat{q} = q_0 + \mathbf{q} \tag{4.2}$$

where q_0 and $\mathbf{q}$ are respectively the scalar and vector parts of the quaternion $\widehat{q}$.

The multiplication rule is a direct consequence of the definition (4.1). In vector notation, the resulting quaternion $\widehat{r}$ can be written

$$\widehat{r} = \widehat{p}\,\widehat{q} = p_0 q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q} \tag{4.3}$$

It is obvious that the quaternion product operation is, in general, non commutative due to the presence of a cross product. commutativity is only achieved when the vector parts of the two operands are parallel as, for example, when one of the operands is a scalar.

The conjugate quaternion to $\widehat{q}$ is defined as

$$\widehat{q}^* = q_0 - iq_1 - jq_2 - kq_3 = q_0 - \mathbf{q} \tag{4.4}$$

It is easily verified that the conjugate of a quaternion product is such that

$$(\widehat{p}\,\widehat{q})^* = \widehat{q}^*\widehat{p}^* \tag{4.5}$$

The norm of a quaternion is calculated by

$$\|\widehat{q}\|^2 = \widehat{q}\,\widehat{q}^* = q_0^2 + \mathbf{q}\cdot\mathbf{q} \tag{4.6}$$

In particular, $\widehat{q}$ is a unit quaternion if

$$\|\widehat{q}\| = 1 \tag{4.7}$$

A quaternion $\widehat{q}$ is a vector quaternion if $\widehat{q} = 0 + \mathbf{q}$, in which case

$$\widehat{q} + \widehat{q}^* = 0 \tag{4.8}$$

4.2 Representation of finite rotations in terms of quaternions

Given a unit quaternion $\widehat{e} = e_0 + \mathbf{e}$ and the position vector $\widehat{x} = 0 + \mathbf{x}$, the finite rotation of $\mathbf{x}$ to a new position $\mathbf{y}$ may be represented by the triple quaternion product

$$\widehat{y} = \widehat{e}\,\widehat{x}\,\widehat{e}^* \tag{4.9}$$

the proof holding by noting that $\widehat{y}$ is also a vector quaternion and that $\|\widehat{y}\| = \|\widehat{x}\|$.

The inverse rotation is directly obtained in terms of the conjugate quaternion

$$\widehat{x} = \widehat{e}^*\,\widehat{y}\,\widehat{e} \tag{4.10}$$

where we have used the fact that $\widehat{e}^*\,\widehat{e} = \|\widehat{e}\|^2 = 1$.

Clearly, every unit quaternion can be expressed in the form:

$$\widehat{e} = \cos\alpha + \mathbf{n}\,\sin\alpha \tag{4.11}$$

where $\mathbf{n}$ is a unit vector. If we perform the operations indicated in (4.10), we obtain:

$$\widehat{e}\,\widehat{x} = -\sin\alpha\,\mathbf{n}\cdot\mathbf{x} + \cos\alpha\,\mathbf{x} + \sin\alpha\,\mathbf{n}\times\mathbf{x} \tag{4.12}$$

The vector character of the result is restored after performing the "symmetric" operation:

$$\begin{aligned}\widehat{y} &= \widehat{e}\,\widehat{x}\,\widehat{e}^* \\ &= \sin^2\alpha\,(\mathbf{n}\cdot\mathbf{x})\,\mathbf{n} + \cos^2\alpha\,\mathbf{x} + 2\,\sin\alpha\,\cos\alpha\,\mathbf{n}\times\mathbf{x} - \sin^2\alpha\,(\mathbf{n}\times\mathbf{x})\times\mathbf{n} \\ &= (\cos 2\alpha\,\mathbf{1} + (1-\cos 2\alpha)\,\mathbf{n}\,\mathbf{n}^T + \sin 2\alpha\,\tilde{\mathbf{n}})\,\mathbf{x}\end{aligned} \tag{4.13}$$

By comparing the latter to equation (2.40), we note that the operation is a rotation of angle 2α about $\mathbf{n}$. In paragraph 3.2, we mentioned two sets of parameters that constitute a unit quaternion: the Euler parameters and the linear parameters. We see that only the Euler parameters allow to represent the rotation operation as a double quaternion product, since any other choice of unit quaternion represents a rotation about $\mathbf{n}$ of angle different from the desired value.

Let now the position vector undergo two successive rotations

$$\begin{aligned}\widehat{y} &= \widehat{e_1}\,\widehat{x}\,\widehat{e_1}^* \\ \widehat{z} &= \widehat{e_2}\,\widehat{x}\,\widehat{e_2}^* = (\widehat{e_2}\widehat{e_1})\,\widehat{x}\,(\widehat{e_2}\widehat{e_1})^*\end{aligned} \tag{4.14}$$

Then, the resulting rotation is given by

$$\hat{z} = \hat{e}\,\hat{x}\,\hat{e}^* \qquad \text{with} \qquad \hat{e} = \hat{e}_2\hat{e}_1 \tag{4.15}$$

It is this property that makes attractive the use of quaternion theory. It allows to easily and economically compute the parameters corresponding to the composed rotation, without resorting to the computation of the rotation operator and subsequent matrix multiplication.

4.3 Matrix representation of quaternions

A quaternion may be represented in matrix form by the 4-dimensional column matrix

$$\hat{\mathrm{q}} = \langle q_0 \quad q_1 \quad q_2 \quad q_3 \rangle^T \tag{4.16}$$

in which case the quaternion product $\hat{a} = \hat{p}\,\hat{q}$ can be written in either form

$$\hat{\mathrm{c}} = \mathbf{A}_p\,\hat{\mathrm{q}} = \mathbf{B}_q\,\hat{\mathrm{p}} \tag{4.17}$$

with the 4×4 matrices

$$\mathbf{A}_p = \begin{bmatrix} p_0 & -\mathrm{p}^T \\ \mathrm{p} & p_0\mathbf{1} + \tilde{\mathrm{p}} \end{bmatrix} \qquad \mathbf{B}_q = \begin{bmatrix} q_0 & -\mathrm{q}^T \\ \mathrm{q} & q_0\mathbf{1} + \tilde{\mathrm{q}} \end{bmatrix} \tag{4.18}$$

where $\mathbf{1}$ is the unit matrix and $\tilde{\mathrm{q}}$ is the skew–symmetric matrix attached to $\mathbf{q}$

$$\tilde{\mathrm{q}}_{ij} = -\epsilon_{ijk}\,q_k \tag{4.19}$$

4.4 Matrix form of rotations by quaternion operations

By using equation (4.14), the rotation operator can be recast in the form

$$\hat{\mathrm{y}} = \mathbf{A}\,\mathbf{B}^T\,\hat{\mathrm{x}} \tag{4.20}$$

When computing the matrix product $\mathbf{AB}^T$ one finds

$$\mathbf{A}\,\mathbf{B}^T = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \tag{4.21}$$

where the 3×3 submatrix $\mathbf{R}$ is the standard rotation operator. Developing the product (4.20) provides the result (3.22):

$$\mathbf{R} = (2e_0{}^2 - 1)\,\mathbf{1} + 2(\mathrm{e}\mathrm{e}^T + e_0\tilde{\mathrm{e}}) \tag{4.22}$$

where $e_0, \mathbf{e}$ are the components of the unit quaternion $\widehat{e}$. This result shows that the quaternion components (e_0, e_1, e_2, e_3) are coincident with the Euler parameters.

Simplified matrix form

From a computational point of view, it is interesting to extract from matrices $\mathbf{A}$ and $\mathbf{B}^T$, the 3×4 sub-matrices:

$$\begin{aligned} \mathbf{H} &= [-\mathbf{e} \quad e_0\mathbf{1} + \tilde{\mathbf{e}}] \\ \mathbf{G} &= [-\mathbf{e} \quad e_0\mathbf{1} - \tilde{\mathbf{e}}] \end{aligned} \tag{4.23}$$

Then, the rotation $\mathbf{R}$ can be expressed as a bilinear form in terms of quaternions:

$$\mathbf{R} = \mathbf{H}\,\mathbf{G}^T \tag{4.24}$$

It is easy to show that the matrices $\mathbf{H}$ and $\mathbf{G}$ verify the relations:

$$\begin{aligned} \mathbf{H}\,\mathbf{H}^T &= \mathbf{G}\,\mathbf{G}^T = \mathbf{1} \\ \mathbf{H}^T\,\mathbf{H} &= \mathbf{G}^T\,\mathbf{G} = \mathbf{1} - \widehat{\mathbf{e}}\,\widehat{\mathbf{e}}^T \\ \mathbf{H}\,\widehat{\mathbf{e}} &= \mathbf{G}\,\widehat{\mathbf{e}} = \mathbf{0} \end{aligned} \tag{4.25}$$

4.5 Different choices of quaternions

In paragraph 3.2, we have introduced four possible choices of quaternions:

- Euler parameters $\widehat{\mathbf{e}} = e_0 + \mathbf{e}$
- Rodrigues parameters $\widehat{\mathbf{b}} = 1 + \mathbf{b}$
- CRV's parameters $\widehat{\mathbf{c}} = c_0 + \mathbf{c}$
- Linear parameters $\widehat{\mathbf{s}} = s_0 + \mathbf{s}$

We have mentioned that although linear parameters constitute a unit quaternion, they do not verify the rule of double quaternion product to express a rotation. The only unit quaternion that verifies this rule is the Euler parameters set.

Rodrigues and CRV parameters can be seen as derived from the Euler parameters by means of a conformal transformation:

$$\begin{aligned} b_i &= \frac{e_i}{e_0} \\ c_i &= \frac{4\,e_i}{1 + e_0} \end{aligned} \tag{4.26}$$

where their respective norms are given by:

$$\begin{aligned} \|\widehat{b}\|^2 &= 1 + \|\mathbf{b}\|^2 \\ \|\widehat{c}\|^2 &= c_0^2 + \|\mathbf{c}\|^2 = (4 - c_0)^2 \end{aligned} \tag{4.27}$$

Then, although they do not form a unit quaternion, they can represent a finite rotation through use of quaternion theory after modifying the expression (4.9) to account for their non-unit length as follows:

$$\begin{aligned} \widehat{y} &= \frac{1}{1+\|\mathbf{b}\|^2}\,\widehat{b}\,\widehat{x}\,\widehat{b}^* \\ \widehat{y} &= \frac{1}{(4-c_0)^2}\,\widehat{c}\,\widehat{x}\,\widehat{c}^* \end{aligned} \tag{4.28}$$

Developing the products (4.28) as in paragraph 4.4, leads to the expression of the rotation operator in terms of the Rodrigues parameters and of CRV's:

$$\begin{aligned} \mathbf{R} &= \frac{1}{(1+\|\mathbf{b}\|^2)}\Big((1-\|\mathbf{b}\|^2)\,\mathbf{1} \;+\; 2(\mathbf{b}\mathbf{b}^T \;+\; \tilde{\mathbf{b}})\Big) \\ \mathbf{R} &= \frac{1}{(4-c_0)^2}\left((c_0^2-\|\mathbf{c}\|^2)\,\mathbf{1} \;+\; 2(\mathbf{c}\mathbf{c}^T \;+\; c_0\tilde{\mathbf{c}})\right) \end{aligned} \tag{4.29}$$

Composition of rotations can be easily and economically performed in terms of Euler parameters, by making use of the quaternion product rule (4.17). The resulting Euler parameters to two partial rotations $\mathbf{e}_1, \mathbf{e}_2$ are next computed:

$$\begin{Bmatrix} e_0 \\ \mathbf{e} \end{Bmatrix} = \begin{Bmatrix} e_{1\,0}e_{2\,0} \;-\; \mathbf{e}_1\cdot\mathbf{e}_2 \\ e_{1\,0}\mathbf{e}_2 \;+\; e_{2\,0}\mathbf{e}_1 \;+\; \mathbf{e}_2\times\mathbf{e}_1 \end{Bmatrix} \tag{4.30}$$

Laws for the composition of rotations in terms of Rodrigues parameters and CRV's can be derived from the quaternion product by modifying the equations to account for their non-unit length. For instance, when working with Rodrigues parameters, we say that the resulting quaternion from the composition of two rotations $\widehat{\mathbf{b}}_1, \widehat{\mathbf{b}}_2$ is given by $\alpha\widehat{\mathbf{b}}_1\widehat{\mathbf{b}}_2$, where α is determined such that the scalar part of the resulting quaternion $\widehat{\mathbf{b}}$ equals 1. From equation (4.3), we have:

$$(\widehat{\mathbf{b}}_1\widehat{\mathbf{b}}_2)_0 \;=\; 1 \;-\; \mathbf{b}_1\cdot\mathbf{b}_2 \tag{4.31}$$

Then, the Rodrigues parameters of the composite rotation are:

$$\mathbf{b} \;=\; \frac{1}{1-\mathbf{b}_1\cdot\mathbf{b}_2}(\mathbf{b}_1 \;+\; \mathbf{b}_2 \;+\; \mathbf{b}_1\times\mathbf{b}_2) \tag{4.32}$$

When working with the conformal rotation vector, the resulting quaternion is computed as $\alpha\widehat{\mathbf{c}}_1\widehat{\mathbf{c}}_2$, where α is now determined so as the scalar part of $\widehat{\mathbf{c}}$ verifies:

$$c_0 \;=\; \frac{16-\|\mathbf{c}\|^2}{8} \tag{4.33}$$

By using equation (4.3) and after imposing this condition, α results

$$\alpha \;=\; \frac{16-\|\mathbf{c}\|^2}{8\left(c_{1\,0}c_{2\,0}-\mathbf{c}_1\cdot\mathbf{c}_2\right)} \tag{4.34}$$

Starting from equation (4.17), and after rather lengthy computation we get the result:

$$\mathbf{c} = \frac{4}{(4-c_{1\,0})(4-c_{2\,0}) + c_{1\,0}c_{2\,0} - \mathbf{c}_1 \cdot \mathbf{c}_2}(c_{1\,0}\mathbf{c}_2 + c_{2\,0}\mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1) \qquad (4.35)$$

5. THE ALGEBRAIC APPROACH: MATRIX ALGEBRA

5.1 The Special Orthogonal group - Parametrization of rotations [9]

Let $SO(3)$ be the (Lie) group of proper orthogonal linear transformations:

$$SO(3) = \{\mathbf{R} : \Re^3 \to \Re^3 \mid \mathbf{R}^T\mathbf{R} = \mathbf{1}\ ,\ \det\mathbf{R} = +1\} \qquad (5.1)$$

Geometrically, each element $\mathbf{R}$ of $SO(3)$ represents a finite rotation, so that $SO(3)$ equipped with the matrix product (or composition of rotations) forms the non commutative group of rotations. The positive sign of the determinant distinguishes rotations from reflections, which are characterized by negatives values of the determinant.

Although $\mathbf{R}$ is a 3×3 matrix , the orthonormality requirement leaves only three free parameters in it. Generally speaking, we may write:

$$\mathbf{R} = \mathbf{R}(\alpha_1, \alpha_2, \alpha_3) \qquad (5.2)$$

where $\alpha_1, \alpha_2, \alpha_3$ are three independent parameters retained to described the rotation. Various choices exist, according to the adopted technique of representation, i.e. Euler angles, Euler parameters, the conformal rotation vector, the rotational vector and Rodrigues parameters, as discussed in paragraph 3.2 . In this section, we will first employ the rotational vector:

$$\boldsymbol{\phi} = \mathbf{n}\,\phi \qquad (5.3)$$

We recall that the exponential map gives the rotation operator in terms of the rotational vector:

$$\mathbf{R} = \mathbf{1} + \widetilde{\boldsymbol{\phi}} + \frac{1}{2!}\widetilde{\boldsymbol{\phi}}^2 + \cdots = \exp(\widetilde{\boldsymbol{\phi}}) \qquad (5.4)$$

where $\widetilde{\boldsymbol{\phi}}$ is the skew-symmetric matrix formed by the components of $\boldsymbol{\phi}$:

$$\begin{aligned} \widetilde{\phi}_{ij} &= -\varepsilon_{ijk}\,\phi_k \\ \phi_i &= -\frac{1}{2}\,\varepsilon_{ijk}\,\widetilde{\phi}_{jk} \end{aligned} \qquad (5.5)$$

5.2 Composite rotations:

Let $\mathbf{R}$ be the rotation operator mapping a rectangular Cartesian frame $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ into the orthonormal frame $\{O'; \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$:

$$\mathbf{t}_I = \mathbf{R}\,\mathbf{E}_I \tag{5.6}$$

Physically, it can be interpreted as a rigid body rotation from the initial to the actual configuration, and $\{O; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\{O'; \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ can be viewed as a material frame and a body-attached or moving frame respectively.

Let us now consider an incremental rotation which carries from the actual frame $\{\mathbf{t}_I\}$ to an updated frame $\{\mathbf{t}'_I\}$. There are two ways of describing this incremental rotation:

i) Via a left translation (spatial rotation)

In this case the total rotation from the initial frame is given by the left-application of an incremental rotation operator $\mathbf{R}_{(l)}$ to the actual rotation $\mathbf{R}$:

$$\begin{aligned} \mathbf{R}' &= \mathbf{R}_{(l)}\,\mathbf{R} \\ \mathbf{t}'_I &= \mathbf{R}_{(l)}\,\mathbf{t}_I \;=\; \mathbf{R}_{(l)}\,\mathbf{R}\,\mathbf{E}_I \end{aligned} \tag{5.7}$$

The incremental rotation can be seen as a rotation applied to the actual frame $\{\mathbf{t}_I\}$.

ii) Via a right translation (material rotation)

Now the total rotation from the initial basis is given by the right-application of an incremental rotation operator $\mathbf{R}_{(r)}$ to the actual rotation $\mathbf{R}$:

$$\begin{aligned} \mathbf{R}' &= \mathbf{R}\,\mathbf{R}_{(r)} \\ \mathbf{t}'_I &= \mathbf{R}\,\mathbf{R}_{(r)}\,\mathbf{E}_I \end{aligned} \tag{5.8}$$

The incremental rotation can be seen as a rotation applied to the material frame $\{\mathbf{E}_I\}$.

Let $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ be the rotational vectors corresponding to the spatial rotation $\mathbf{R}_{(l)}$ and to the material rotation $\mathbf{R}_{(r)}$, respectively:

$$\begin{aligned} \mathbf{R}_{(l)} &= \exp(\widetilde{\boldsymbol{\theta}}) \\ \mathbf{R}_{(r)} &= \exp(\widetilde{\boldsymbol{\Theta}}) \end{aligned} \tag{5.9}$$

Using equations (5.6-7), it can be easily seen that the spatial and the material incremental vectors are mutually related by:

$$\boldsymbol{\theta} = \mathbf{R}\,\boldsymbol{\Theta} \tag{5.10}$$

Other parametrization techniques

The formulas developed in this paragraph using the rotational vector, have a correspondence with similar relations obtained by using the other parametrization techniques. We can define spatial and material versions of the angular increments for the other parametrizations that we have discussed in paragraph 3.2:

$$\begin{aligned} \mathbf{R}_{(l)} &= \exp(\widetilde{\boldsymbol{\theta}}) = \mathbf{f}_1(\mathbf{b}) = \mathbf{f}_2(\mathbf{e}) = \mathbf{f}_3(\mathbf{c}) \\ \mathbf{R}_{(r)} &= \exp(\widetilde{\boldsymbol{\Theta}}) = \mathbf{f}_1(\mathbf{B}) = \mathbf{f}_2(\mathbf{E}) = \mathbf{f}_3(\mathbf{C}) \end{aligned} \tag{5.11}$$

where $\mathbf{f}_1(\ ), \mathbf{f}_2(\ ), \mathbf{f}_3(\ )$ were explicited in equations (3.14), (3.22) and (3.30), and where $(\mathbf{b}, \mathbf{B})$, $(\mathbf{e}, \mathbf{E})$, and $(\mathbf{c}, \mathbf{C})$ denote the spatial and material angular increments in terms of Rodrigues, Euler and CRV parameters, respectively.

By noting that $\mathbf{R}_{(l)} = \mathbf{R}\,\mathbf{R}_{(r)}\,\mathbf{R}^T$ and by using equations (3.14),(3.22) and (3.30), it can be shown that equation (5.9) which relates material and spatial increments, also holds for these parametrizations:

$$\begin{aligned} \mathbf{b} &= \mathbf{R}\,\mathbf{B} \\ \mathbf{e} &= \mathbf{R}\,\mathbf{E} \qquad (e_0 = E_0) \\ \mathbf{c} &= \mathbf{R}\,\mathbf{C} \end{aligned} \tag{5.12}$$

The expression of the composition of rotations in terms of the quaternion product corresponds to the left translation composition. Right translation updating is simply obtained by commuting the operands, that is to say, by expressing the composition as the (quaternion) product of the parameters corresponding to $\mathbf{R}$ times the parameters corresponding to $\mathbf{R}_{(r)}$. For instance, if $\mathbf{b}_1$ are the Rodrigues parameters of the actual rotation and $\mathbf{b}$ and $\mathbf{B}$ are the Rodrigues parameters of the spatial and material increments, the parameters $\mathbf{b}_1'$ of the updated rotation are:

$$\begin{aligned} \left\{\begin{matrix} 1 \\ \mathbf{b}_1' \end{matrix}\right\} &= \widehat{\mathbf{b}}\,\widehat{\mathbf{b}}_1 = \left\{\begin{matrix} 1 \\ \frac{1}{1-\mathbf{b}\cdot\mathbf{b}_1}(\mathbf{b} + \mathbf{b}_1 + \mathbf{b}\times\mathbf{b}_1) \end{matrix}\right\} \\ \left\{\begin{matrix} 1 \\ \mathbf{b}_1' \end{matrix}\right\} &= \widehat{\mathbf{b}}_1\,\widehat{\mathbf{B}} = \left\{\begin{matrix} 1 \\ \frac{1}{1-\mathbf{b}_1\cdot\mathbf{B}}(\mathbf{b}_1 + \mathbf{B} + \mathbf{b}_1\times\mathbf{B}) \end{matrix}\right\} \end{aligned} \tag{5.13}$$

5.3 Derivatives of the rotation operator

Linearized incremental rotations are given by the application of the *directional (Fréchet) derivative* to the rotation operator. They consist thus on skew-symmetric matrices applied to the actual rotation:

$$\begin{aligned} D\mathbf{R}\cdot\widetilde{\boldsymbol{\theta}} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\widetilde{\boldsymbol{\theta}})\,\mathbf{R} = \widetilde{\boldsymbol{\theta}}\,\mathbf{R} \\ D\mathbf{R}\cdot\widetilde{\boldsymbol{\Theta}} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{R}\,\exp(\varepsilon\widetilde{\boldsymbol{\Theta}}) = \mathbf{R}\,\widetilde{\boldsymbol{\Theta}} \end{aligned} \tag{5.14}$$

The skew-symmetric tensor $\widetilde{\boldsymbol{\theta}}$ ($\widetilde{\boldsymbol{\Theta}}$) represents infinitesimal or linearized spatial (material) incremental rotations about the eigenvector $\boldsymbol{\theta}$ ($\boldsymbol{\Theta}$) associated with the only zero eigenvalue:

$$\widetilde{\boldsymbol{\theta}}\,\boldsymbol{\theta} = \mathbf{0} \quad ; \quad \widetilde{\boldsymbol{\Theta}}\,\boldsymbol{\Theta} = \mathbf{0} \tag{5.15}$$

The axial vector $\boldsymbol{\theta}$ ($\boldsymbol{\Theta}$) and the skew-symmetric tensor $\widetilde{\boldsymbol{\theta}}$ ($\widetilde{\boldsymbol{\Theta}}$) are mutually related by equations (5.4). The set of skew-symmetric matrices form the linear vector space $so(3)$:

$$so(3) = \{\widetilde{\boldsymbol{\Theta}} \mid \widetilde{\boldsymbol{\Theta}} + \widetilde{\boldsymbol{\Theta}}^T = 0\} \tag{5.16}$$

This space is isomorphic to $\Re^3$, the isomorphism being defined by equation (5.4).

Given any $\mathbf{R} \in SO(3)$ and any $\boldsymbol{\Theta} \in so(3)$, linearized incremental rotations constitute the tangent space $T_{\mathbf{R}}SO(3)$ at a point $\mathbf{R} \in SO(3)$, which may be represented in two alternative forms

i) Left invariant vector fields defined by the left application of the increment to the actual rotation. Accordingly we set

$$^{l}T_{\mathbf{R}}SO(3) = \{\widetilde{\boldsymbol{\theta}}\,\mathbf{R} \quad \forall \quad \widetilde{\boldsymbol{\theta}} \in so(3)\} \tag{5.17}$$

Geometrically, an element $\widetilde{\boldsymbol{\theta}}\mathbf{R} \in {}^{l}T_{\mathbf{R}}SO(3)$ corresponds to an infinitesimal rotation $\widetilde{\boldsymbol{\theta}} \in so(3)$ superposed onto a finite rotation $\mathbf{R}$. Following standard usage, we define the variation of rotations expressed in terms of spatial components of angular variations, as follows:

$$\delta\mathbf{R} = \delta\widetilde{\boldsymbol{\theta}}\,\mathbf{R} \tag{5.18}$$

ii) Right invariant vector fields. They are characterized as in (i), but with a right translation instead of a left translation:

$$T_{\mathbf{R}}SO(3) = \{\mathbf{R}\,\widetilde{\boldsymbol{\Theta}} \quad \forall \quad \widetilde{\boldsymbol{\Theta}} \in so(3)\} \tag{5.19}$$

Geometrically, an element $\mathbf{R}\widetilde{\boldsymbol{\Theta}} \in T_{\mathbf{R}}SO(3)$ corresponds to a finite rotation superposed onto an infinitesimal rotation $\widetilde{\boldsymbol{\Theta}} \in so(3)$. The variation of rotations is now accordingly defined in terms of material components of angular variations:

$$\delta\mathbf{R} = \mathbf{R}\,\delta\widetilde{\boldsymbol{\Theta}} \tag{5.20}$$

We will say that $so(3)$ forms the tangent space of $SO(3)$ at the identity $\mathbf{1} \in SO(3)$, and we employ the notation $so(3) = T_{\mathbf{1}}SO(3)$. Obviously, no distinction is made in this case between left and right applications of the increment.

The tangent space $T_{\mathbf{R}}SO(3)$ is isomorphic to $\Re^3$ and so, we can talk of linearized rotation increments (the axial vectors $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$) as vectors in $\Re^3$:

$$T_{\mathbf{R}}SO(3) \sim so(3) \sim \Re^3 \tag{5.21}$$

Other parametrization techniques

Again, we are able to establish a correlation with the other parametrization techniques. Linearized increments are given by the application of the Fréchet derivative to the rotation operator, now in terms of the employed parameters, giving:

$$
\begin{aligned}
D\mathbf{R}\cdot\mathbf{b} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{f}_1(\varepsilon\mathbf{b})\,\mathbf{R} &&= 2\,\tilde{\mathbf{b}}\,\mathbf{R} \\
D\mathbf{R}\cdot\mathbf{B} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{R}\,\mathbf{f}_1(\varepsilon\mathbf{B}) &&= 2\,\mathbf{R}\,\tilde{\mathbf{B}} \\
D\mathbf{R}\cdot\mathbf{e} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{f}_2(\varepsilon\mathbf{e})\,\mathbf{R} &&= 2\,\tilde{\mathbf{e}}\,\mathbf{R} \\
D\mathbf{R}\cdot\mathbf{E} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{R}\,\mathbf{f}_2(\varepsilon\mathbf{E}) &&= 2\,\mathbf{R}\,\tilde{\mathbf{E}} \\
D\mathbf{R}\cdot\mathbf{c} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{f}_3(\varepsilon\mathbf{c})\,\mathbf{R} &&= \tilde{\mathbf{c}}\,\mathbf{R} \\
D\mathbf{R}\cdot\mathbf{C} &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{R}\,\mathbf{f}_3(\varepsilon\mathbf{C}) &&= \mathbf{R}\,\tilde{\mathbf{C}}
\end{aligned}
\tag{5.22}
$$

Now, the skew-symmetric tensors $\tilde{\mathbf{b}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{e}}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{C}}$ give the linearized incremental rotations with respect to the current rotation $\mathbf{R} \in SO(3)$. Since Rodrigues and Euler parameters are a function of $(\phi/2)$, there appears a factor 2 in their expressions. We recall that the CRV's definition was corrected so that its variation coincides with the angle variation.

We are able to define tangent spaces to $SO(3)$ at $\mathbf{R}$, formed by the set of linearized increments in each parametrization technique. These tangent spaces can be seen as originated by using different internal products than that we have (implicitly) used to build ${}^{l}T_{\mathbf{R}}SO(3)$ and $T_{\mathbf{R}}SO(3)$.

Straight lines in each tangent space generate curves on $SO(3)$. These curves are geodesic lines with respect to an appropriate measure, which is related to the internal product associated to each parametrization technique.

5.4 Relation between increments to two different rotation operators

Rotations are objects belonging to a nonlinear manifold, the so-called special orthogonal Lie group $SO(3)$. Since $SO(3)$ does not form a vector space, certain operations are not allowed in it (i.e. interpolation). In order to make computations we work with the vector space $T_{\mathbf{R}}SO(3)$, the space of rotation increments with respect to a given rotation $\mathbf{R}$. Interpolation in $SO(3)$ will then be defined in terms of the interpolation in $T_{\mathbf{R}}SO(3)$.

Let us address a problem related to the operation of interpolation: we want to determine the relation between linearized rotations at the tangent space to $SO(3)$ at $\mathbf{R}_{(A)}$ and

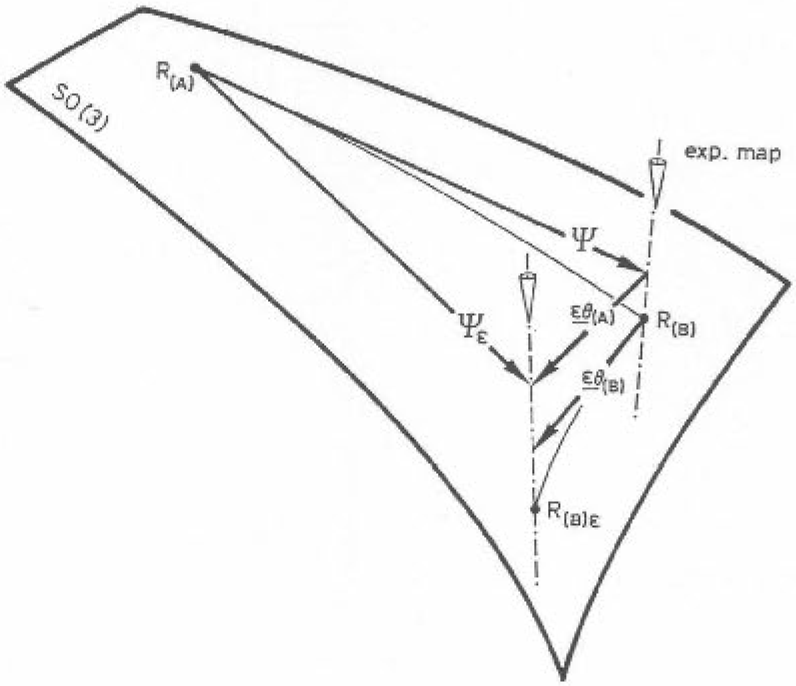

Figure 13: Projection of the rotation increment

linearized rotations belonging to the tangent space to $SO(3)$ at a second point $\mathbf{R}_{(\mathbf{B})}$.

Let $\mathbf{R}_{(\mathbf{A})}, \mathbf{R}_{(\mathbf{B})} \in SO(3)$ be two given rotations located such that:

$$\mathbf{R}_{(\mathbf{B})} = \mathbf{R}_{(\mathbf{A})} \exp(\widetilde{\mathbf{\Psi}}) \tag{5.23}$$

where:

$$\mathbf{R}_{(\mathbf{A})} \widetilde{\mathbf{\Psi}} \in T_{\mathbf{R}_{(\mathbf{A})}} SO(3)$$

Let $\mathbf{R}_{(\mathbf{B})\varepsilon}$ be the perturbed rotation at $\mathbf{R}_{(\mathbf{B})}$:

$$\mathbf{R}_{(\mathbf{B})\varepsilon} = \mathbf{R}_{(\mathbf{B})} \exp(\varepsilon \widetilde{\mathbf{\Theta}}_{(\mathbf{B})}) \tag{5.24}$$

where $\mathbf{R}_{(\mathbf{B})} \widetilde{\mathbf{\Theta}}_{(\mathbf{B})} \in T_{\mathbf{R}_{(\mathbf{B})}} SO(3)$. Its axial vector is denoted as $\mathbf{\Theta}_{(\mathbf{B})}$ in order to remind the vector space to which it belongs.

$\mathbf{R}_{(\mathbf{B})\varepsilon}$ can also be expressed in terms of increments belonging to the tangent space at $\mathbf{R}_{(\mathbf{A})}$:

$$\mathbf{R}_{(\mathbf{B})\varepsilon} = \mathbf{R}_{(\mathbf{A})} \exp(\widetilde{\mathbf{\Psi}}_\varepsilon) \tag{5.25}$$

where

$$\mathbf{\Psi}_\varepsilon = \mathbf{\Psi} + \varepsilon \mathbf{\Theta}_{(\mathbf{A})}$$

and where $\mathbf{\Theta}_{(\mathbf{A})}$ gives linearized increments in $T_{\mathbf{R}_{(\mathbf{A})}} SO(3)$. Note that the addition has meaning since $\mathbf{\Psi}$ and $\mathbf{\Theta}_{(\mathbf{A})}$ belong to the same vector space.

Let us now determine the relation existing between $\mathbf{\Theta}_{(\mathbf{A})}$ and $\mathbf{\Theta}_{(\mathbf{B})}$ (We note that both vectors refer to the same object, the rotation increment from $\mathbf{R}_{(\mathbf{B})}$ to $\mathbf{R}_{(\mathbf{B})\varepsilon}$). The situation is illustrated in figure 13.

By equating (5.23) and (5.24), we get the expression:

$$\exp(\varepsilon\widetilde{\Theta}_{(\mathbf{B})}) = \exp(-\widetilde{\Psi}) \exp(\widetilde{\Psi}_\varepsilon) \tag{5.26}$$

The latter equation, in fact, indicates that the rotation represented by $\varepsilon\Theta_{(\mathbf{B})}$ results from the composition of two rotations: one of axial vector $-\Psi$ and the other with axial vector Ψ_ε.

We have mentioned that the composition rule is rather difficult to express in terms of rotational vectors, but it is a simple algebraic expression when given in terms of Rodrigues parameters. Let $\overline{(\ )}$ be the transformation that gives the Rodrigues parameters in terms of the rotational vector:

$$\begin{aligned} \overline{(\ )} &: \{Rot.vect.\} \rightarrow \{Rodrigues\ param.\} \\ \overline{\Psi} &= \frac{\tan(\|\Psi\|/2)}{\|\Psi\|}\ \Psi \end{aligned} \tag{5.27}$$

where $\|\Psi\| = (\Psi \cdot \Psi)^{\frac{1}{2}}$.

Then, we apply the law for the composition of rotations in terms of Rodrigues parameters (equation (4.32)), and we rewrite equation (5.25) to get:

$$\overline{\varepsilon\Theta_{(\mathbf{B})}} = \frac{1}{1 + \overline{\Psi} \cdot \overline{\Psi}_\varepsilon} \left[\overline{\Psi}_\varepsilon - \overline{\Psi} - \overline{\Psi}_\varepsilon \times \overline{\Psi}\right] \tag{5.28}$$

Next, we will compute the derivative of equation (5.28) with respect to ε at $\varepsilon = 0$. By differentiating equation (5.26), we obtain:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{\varepsilon\Theta_{(\mathbf{B})}} = \frac{\Theta_{(\mathbf{B})}}{2} \tag{5.29}$$

The differentiation of the right-hand-side of equation (5.28) leads to the expression:

$$\frac{\Theta_{(\mathbf{B})}}{2} = \frac{1}{1 + \|\overline{\Psi}\|^2} \left\{\mathbf{1} + [\overline{\Psi}\times]\right\} D\overline{\Psi} \cdot \Theta_{(\mathbf{A})} \tag{5.30}$$

The Fréchet differential of $\overline{\Psi}$ is computed as follows:

$$D\overline{\Psi} \cdot \Theta_{(\mathbf{A})} = \left[\frac{\tan(\|\Psi\|/2)}{\|\Psi\|} (\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) + \frac{1}{2\cos^2(\|\Psi\|/2)}\, \mathbf{n} \otimes \mathbf{n}\right] \Theta_{(\mathbf{A})} \tag{5.31}$$

where the unit vector $\mathbf{n}$ is defined as:

$$\mathbf{n} = \frac{\Psi}{\|\Psi\|}$$

Finally, by replacing equations (5.31) and (5.26) into equation (5.30), we obtain the result:

$$\Theta_{(\mathbf{B})} = \mathbf{T}(\Psi)\ \Theta_{(\mathbf{A})} \tag{5.32}$$

with the linear transformation $\mathbf{T}(\boldsymbol{\Psi})$ defined by:

$$\mathbf{T}(\boldsymbol{\Psi}) \;=\; \frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\,\mathbf{1} \;+\; \left(1-\frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\right)\mathbf{n}\otimes\mathbf{n} \;-\; \frac{1}{2}\left(\frac{\sin(\|\boldsymbol{\Psi}\|/2)}{(\|\boldsymbol{\Psi}\|/2)}\right)^2\,\tilde{\boldsymbol{\Psi}} \qquad (5.33)$$

It is clearly seen that when $\boldsymbol{\Psi}\rightarrow\mathbf{0}\Longrightarrow\mathbf{T}(\boldsymbol{\Psi})\rightarrow\mathbf{1}$, as expected.

The inverse relation can be easily obtained by using equation (3.11). After some algebraic steps we arrive at the final result:

$$\boldsymbol{\Theta}_{(\mathbf{A})} = \mathbf{T}^{-1}(\boldsymbol{\Psi})\,\boldsymbol{\Theta}_{(\mathbf{B})} \qquad (5.34)$$

with the linear transformation $\mathbf{T}^{-1}(\boldsymbol{\Psi})$ given by:

$$\mathbf{T}^{-1}(\boldsymbol{\Psi}) \;=\; \frac{\|\boldsymbol{\Psi}\|/2}{\tan(\|\boldsymbol{\Psi}\|/2)}\,\mathbf{1} \;+\; \left(1-\frac{\|\boldsymbol{\Psi}\|/2}{\tan(\|\boldsymbol{\Psi}\|/2)}\right)\mathbf{n}\otimes\mathbf{n} \;+\; \frac{1}{2}\,\tilde{\boldsymbol{\Psi}} \qquad (5.35)$$

5.5 Expression of angular variations in terms of variations of the rotation parameters

Equation (5.32-33) can be particularized to the case in which one relates incremental angle variations and rotation parameters variations. Clearly, if the rotation operator is parametrized by using the rotational vector, equation (5.32) states that:

$$\delta\boldsymbol{\Theta} \;=\; \mathbf{T}(\boldsymbol{\Psi})\;\delta\boldsymbol{\Psi} \qquad (5.36)$$

Similar relations can be established for the other techniques of parametrization. In particular, when working with Rodrigues parameters, equation (5.30) shows that:

$$\delta\boldsymbol{\Theta} \;=\; \mathbf{T}(\mathbf{B})\;\delta\mathbf{B} \qquad (5.37)$$

with

$$\mathbf{T}(\mathbf{B}) \;=\; \frac{2}{1+\|\mathbf{B}\|^2}\,[\mathbf{1}\;+\;\tilde{\mathbf{B}}] \qquad (5.38)$$

Expressions for Euler parameters and CRV's can be obtained after replacing the relations between Rodrigues parameters and Euler's and CRV's into (5.37), giving:

$$\delta\boldsymbol{\Theta} \;=\; \mathbf{T}(E_0,\mathbf{E})\begin{Bmatrix}\delta E_0\\ \delta\mathbf{E}\end{Bmatrix} \qquad (5.39)$$

with the 3×4 matrix:

$$\mathbf{T}(E_0,\mathbf{E}) \;=\; 2\;[-\mathbf{E}\quad E_0\mathbf{1}\;+\;\tilde{\mathbf{E}}] \qquad (5.40)$$

and giving:

$$\delta\boldsymbol{\Theta} \;=\; \mathbf{T}(\mathbf{C})\;\delta\mathbf{C} \qquad (5.41)$$

with

$$\mathbf{T}(\mathbf{C}) = \frac{1}{(4 - C_0)^2} [C_0 \mathbf{1} + \tilde{\mathbf{C}} - \frac{\mathbf{C}\mathbf{C}^T}{4}] \tag{5.42}$$

The spatial components of the rotation variations $\delta\boldsymbol{\theta}$ are related to the material variations $\delta\boldsymbol{\Theta}$ through the rotation operator $\mathbf{R}$ in the form:

$$\delta\boldsymbol{\theta} = \mathbf{R}\,\delta\boldsymbol{\Theta} \tag{5.43}$$

The spatial variations of the rotation parameters $\delta\boldsymbol{\psi}$ are:

$$\delta\boldsymbol{\psi} = \mathbf{R}\,\delta\boldsymbol{\Psi} \tag{5.44}$$

Let us also note the following property of $\mathbf{T}$:

$$\mathbf{R}\,\mathbf{T}(\boldsymbol{\Psi})\,\mathbf{R}^T = \mathbf{R} \tag{5.45}$$

This property can be shown easily by replacing equation (5.33) into it and by noting that $\mathbf{R}\mathbf{n} = \mathbf{n}$. Then, the spatial components of rotation variations are computed in terms of the spatial components of the parameters variation as follows:

$$\delta\boldsymbol{\theta} = \mathbf{T}(\boldsymbol{\psi})\,\delta\boldsymbol{\psi} \tag{5.46}$$

where we have used the fact that $\boldsymbol{\Psi} = \boldsymbol{\psi}$.

We remark that the material and the spatial rotational vectors differ only in the way increments are handled; so, although their values coincide for a given rotation, their respective increments are not coincident but related through the rotation operator. We will see that the same holds for velocities and accelerations, which can be seen as increments.

Similar expressions are developed for the other techniques of parametrization. It is interesting to note that equation (5.45) is valid for all of them with the only exception of the Euler parameters, which takes the form:

$$\mathbf{R}\,\mathbf{T}(e_0, \mathbf{e}) \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}^T \end{bmatrix} = \mathbf{T}(e_0, \mathbf{e}) \tag{5.47}$$

Thus, the final expressions we obtain are:

$$\begin{aligned} \delta\boldsymbol{\theta} &= \mathbf{T}(\mathbf{b})\,\delta\mathbf{b} \\ &= \mathbf{T}(e_0, \mathbf{e}) \begin{Bmatrix} \delta e_0 \\ \delta\mathbf{e} \end{Bmatrix} \\ &= \mathbf{T}(\mathbf{c})\,\delta\mathbf{c} \end{aligned} \tag{5.48}$$

5.6 Range of validity of the different parametrizations

By analyzing the definitions of the Rodrigues and CRV parameters, we see that they are limited to values of ϕ in the intervals $(-\pi, \pi)$ and $(-2\pi, 2\pi)$, respectively (see equations (3.19) and (3.26)). At the extremes of these intervals they present a singularity, when the tangent trigonometric function goes to infinity.

The rotational vector does not show any singularity in its definition, and it can go up to any magnitude of rotation. However, it evidences a "hole in differentiability" for values of $\phi = \pm 2\pi$, in which the matrix $\mathbf{T}(\mathbf{\Psi})$ becomes singular (see equations (5.33-35)). This fact precludes the direct employment of the rotational vector for rotations of arbitrary magnitude. We note here in pass, that a similar analysis for the CRV parameters limits further their range of application to the interval $\phi \in (-3.709, 3.709)$.

The Euler parameters are the only set that allow to treat rotations of arbitrary magnitude, without resorting to any special trick. They do not possess any limitation from the point of view of their definition, and equation (5.39), which gives the angles variation in term of the parameters variation, can be easily inverted for any value of rotation:

$$\left\{ \begin{array}{c} \delta E_0 \\ \delta \mathbf{E} \end{array} \right\} = \frac{1}{2} \left[\begin{array}{c} -\mathbf{E}^T \\ E_0 \mathbf{1} - \widetilde{\mathbf{E}} \end{array} \right] \delta \Theta \tag{5.49}$$

The latter expression is obtained using equations (4.25).

The inconvenience of the limitations in the magnitude of rotations can be surmounted, for the rotational vector and the CRV parameters, by working only with values in the range $|\phi| \leq \pi$. Clearly, this range is enough to represent any rotation in three dimensional space; rotations exceeding this range can be transformed to fall into this class. The interest for doing this trick is to allow to use a parametrization with a minimal set of parameters, and consequently minimize the number of degrees of freedom of the discretization.

The limitation $|\phi| \leq \pi$ is translated in terms of parameters:

$$\begin{aligned} \|\mathbf{\Psi}\| &\leq \pi \\ \|\mathbf{c}\| &\leq 4 \end{aligned} \tag{5.50}$$

Whenever any rotation violates condition (5.50-a), the rotational vector is corrected according to:

$$\mathbf{\Psi}^* = \left(1 - \frac{2\pi}{\|\mathbf{\Psi}\|}\right) \mathbf{\Psi} \tag{5.51}$$

It can be readily verified that $\mathbf{\Psi}^*$ satisfies the two following conditions:

$$\text{i)} \qquad \mathbf{R}(\mathbf{\Psi}^*) = \mathbf{R}(\mathbf{\Psi}) \qquad\qquad (\exp(\widetilde{\mathbf{\Psi}}^*) = \exp(\widetilde{\mathbf{\Psi}}))$$

$$\text{ii)} \qquad \pi \leq \|\mathbf{\Psi}\| \leq 2\pi \quad \Rightarrow \quad \|\mathbf{\Psi}^*\| \leq \pi$$

A similar reasoning leads to the following formulas for the CRV parameters:

$$\mathbf{C}^* = -\left(\frac{16}{\|\mathbf{C}\|^2}\right) \mathbf{C} \tag{5.52}$$

where $\mathbf{C}^*$ verifies the conditions:

$$\mathbf{R}(\mathbf{C}^*) = \mathbf{R}(\mathbf{C})$$

$$\|\mathbf{C}\| \geq 4 \quad \Rightarrow \quad \|\mathbf{C}^*\| \leq 4$$

6. ANGULAR VELOCITIES AND ACCELERATIONS

6.1 Spatial and material angular velocities and accelerations

The tensor of angular velocities gives the variation in time of rotations. It can be expressed in terms of spatial or material components of rotations, as we did for the incremental rotations:

$$\begin{aligned} \widetilde{\omega} &= \frac{d\mathbf{R}}{dt}\,\mathbf{R}^T \\ \widetilde{\mathbf{\Omega}} &= \mathbf{R}^T\,\frac{d\mathbf{R}}{dt} \end{aligned} \tag{6.1}$$

where $\widetilde{\omega}$ and $\widetilde{\mathbf{\Omega}}$ represent the skew-symmetric tensors of spatial and material angular velocities. The corresponding axial vectors $\boldsymbol{\omega}$ and $\mathbf{\Omega}$, related to them by equation (5.4), are the spatial and material vectors of angular velocities.

The skew-symmetric tensors of spatial and material angular accelerations are defined by time-differentiating the angular velocities:

$$\begin{aligned} \widetilde{\alpha} &= \dot{\widetilde{\omega}} = \dot{\mathbf{R}}\,\dot{\mathbf{R}}^T + \ddot{\mathbf{R}}\,\mathbf{R}^T \\ \widetilde{\mathbf{A}} &= \dot{\widetilde{\mathbf{\Omega}}} = \dot{\mathbf{R}}^T\,\dot{\mathbf{R}} + \mathbf{R}^T\,\ddot{\mathbf{R}} \end{aligned} \tag{6.2}$$

The angular accelerations $\boldsymbol{\alpha}$ and $\mathbf{A}$ are given by the corresponding axial vectors.

The objective of this section is to give the explicit expression of the angular velocities and accelerations as a function of the selected parametrization technique. We will separate the treatment of non-invariant parametrizations (i.e. Euler angles) from that of invariant ones (i.e. the rotational vectors). In the first case, the velocities expressions will be developed based on geometrical considerations, while in the second case we will make use of the relation (5.32), which was built from an algebraic point of view.

6.2 Non-invariant parametrizations

Euler angles

One could proceed to a direct computation by differentiation of the rotation operator and subsequent application of the equation (6.1). However, it is more simple (and elegant) to make the following geometric reasoning.

In terms of Euler angles, the velocities vector results from three superposed motions around the three following intermediate coordinate axes (Figure 14):

- a rotation around Oz with velocity $\dot{\phi}$.
- a rotation around Ox_1 with velocity $\dot{\theta}$.
- a rotation around Oz_2 with velocity $\dot{\psi}$. Then, we have:

$$\omega = \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \mathbf{R}(z,\phi) \begin{Bmatrix} \dot{\theta} \\ 0 \\ 0 \end{Bmatrix} + \mathbf{R}(z,\phi)\mathbf{R}(x_1,\theta) \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix} \tag{6.3}$$

After making the computations, one finds the expression of the angular velocity:

$$\omega = \begin{bmatrix} 0 & \cos\phi & \sin\phi\sin\theta \\ 0 & \sin\phi & -\cos\phi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} \tag{6.4}$$

where $\dot{\phi}$ $\dot{\theta}$ $\dot{\psi}$ are the time derivatives of the rotation parameters.

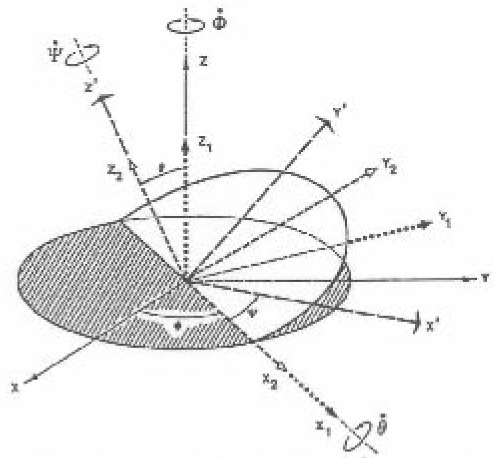

Figure 14: Angular velocities in terms of Euler angles

Bryant angles

With a similar reasoning to the preceding case, we can decompose the angular velocity in the form:

- a rotation around Oz with velocity $\dot{\psi}$.
- a rotation around Oy_1 with velocity $\dot{\theta}$.
- a rotation around Ox_2 with velocity $\dot{\phi}$.

After making the computations, one finds:

$$\omega = \begin{bmatrix} 0 & -\sin\psi & \cos\psi\cos\theta \\ 0 & \cos\psi & \sin\psi\cos\theta \\ 1 & 0 & -\sin\theta \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \tag{6.5}$$

where $\dot{\psi}$ $\dot{\theta}$ $\dot{\phi}$ are the time derivatives of the Bryant angles.

6.3 Invariant parametrizations

We have shown in paragraph 5.5 that angular variations are related to the variation of the rotation parameters by the matrix $\mathbf{T}$, and we also gave the expression of $\mathbf{T}$ in the different parametrizations. If we now consider that this variation is made in a time differential, it is easy to recognize in (5.36) the expression for the material angular velocities:

$$\boldsymbol{\Omega} = \mathbf{T}(\boldsymbol{\Psi})\,\dot{\boldsymbol{\Psi}} \tag{6.6}$$

The latter equation is stated in terms of the (material) rotational vector and its time derivative. When working with the other parametrizations, similar expressions can be developed:

$$\begin{aligned} \boldsymbol{\Omega} &= \mathbf{T}(\mathbf{B})\,\dot{\mathbf{B}} \\ &= \mathbf{T}(E_0,\mathbf{E}) \begin{Bmatrix} \dot{E}_0 \\ \dot{\mathbf{E}} \end{Bmatrix} \\ &= \mathbf{T}(\mathbf{C})\,\dot{\mathbf{C}} \end{aligned} \tag{6.7}$$

Spatial angular velocities are related to the material ones through the rotation operator ($\omega = \mathbf{R}\Omega$). By using, as in paragraph 5.5, the fact that $\mathbf{R}\,\mathbf{T}\,\mathbf{R}^T = \mathbf{T}$, we obtain the result:

$$\omega = \mathbf{T}(\psi)\,\dot{\psi} \tag{6.8}$$

The same can be done for the other parametrizations, giving:

$$\begin{aligned} \omega &= \mathbf{T}(\mathbf{b})\,\dot{\mathbf{b}} \\ &= \mathbf{T}(e_0,\mathbf{e}) \begin{Bmatrix} \dot{e}_0 \\ \dot{\mathbf{e}} \end{Bmatrix} \\ &= \mathbf{T}(\mathbf{c})\,\dot{\mathbf{c}} \end{aligned} \tag{6.9}$$

The angular accelerations are computed by time-differentiating equations (6.6) or (6.8), depending on whether material or spatial components of rotations are employed:

$$\begin{aligned} \mathbf{A} &= \mathbf{T}(\boldsymbol{\Psi})\,\ddot{\boldsymbol{\Psi}} + \dot{\mathbf{T}}(\boldsymbol{\Psi})\,\dot{\boldsymbol{\Psi}} \\ \alpha &= \mathbf{T}(\psi)\,\ddot{\psi} + \dot{\mathbf{T}}(\psi)\,\dot{\psi} \end{aligned} \tag{6.10}$$

Similarly, for the other parametrization techniques, we get

$$\begin{aligned}
\mathbf{A} &= \mathbf{T}(\mathbf{B})\,\ddot{\mathbf{B}} + \dot{\mathbf{T}}(\mathbf{B})\,\dot{\mathbf{B}} \\
&= \mathbf{T}(E_0,\mathbf{E}) \begin{Bmatrix} \ddot{E}_0 \\ \ddot{\mathbf{E}} \end{Bmatrix} + \dot{\mathbf{T}}(E_0,\mathbf{E}) \begin{Bmatrix} \dot{E}_0 \\ \dot{\mathbf{E}} \end{Bmatrix} \\
&= \mathbf{T}(\mathbf{C})\,\ddot{\mathbf{C}} + \dot{\mathbf{T}}(\mathbf{C})\,\dot{\mathbf{C}} \\
\alpha &= \mathbf{T}(\mathbf{b})\,\ddot{\mathbf{b}} + \dot{\mathbf{T}}(\mathbf{b})\,\dot{\mathbf{b}} \\
&= \mathbf{T}(e_0,\mathbf{e}) \begin{Bmatrix} \ddot{e}_0 \\ \ddot{\mathbf{e}} \end{Bmatrix} + \dot{\mathbf{T}}(e_0,\mathbf{e}) \begin{Bmatrix} \dot{e}_0 \\ \dot{\mathbf{e}} \end{Bmatrix} \\
&= \mathbf{T}(\mathbf{c})\,\ddot{\mathbf{c}} + \dot{\mathbf{T}}(\mathbf{c})\,\dot{\mathbf{c}}
\end{aligned} \tag{6.11}$$

We will see, after computing $\dot{\mathbf{T}}(E_0,\mathbf{E})$, that the acceleration equations in terms of Euler parameters are directly simplified to

$$\begin{aligned}
\mathbf{A} &= \mathbf{T}(E_0,\mathbf{E}) \begin{Bmatrix} \ddot{E}_0 \\ \ddot{\mathbf{E}} \end{Bmatrix} \\
\alpha &= \mathbf{T}(e_0,\mathbf{e}) \begin{Bmatrix} \ddot{e}_0 \\ \ddot{\mathbf{e}} \end{Bmatrix}
\end{aligned} \tag{6.12}$$

The expressions of $\dot{\mathbf{T}}$ for the different parametrizations are developed in the next subsection.

Derivatives of $\mathbf{T}$

The Fréchet derivative of $\mathbf{T}(\boldsymbol{\Psi})$ will be first evaluated, by differentiating equation (5.33), giving:

$$\begin{aligned}
D\mathbf{T}\cdot\Delta\boldsymbol{\Psi} = &\left(\cos\|\boldsymbol{\Psi}\| - \frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\right)\frac{\mathbf{n}\cdot\Delta\boldsymbol{\Psi}}{\|\boldsymbol{\Psi}\|}\,\mathbf{1} + \left(1 - \frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\right)\left[\frac{\Delta\boldsymbol{\Psi}\otimes\mathbf{n} + \mathbf{n}\otimes\Delta\boldsymbol{\Psi}}{\|\boldsymbol{\Psi}\|}\right] \\
&+ \left(3\,\frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|} - \cos\|\boldsymbol{\Psi}\| - 2\right)\frac{\mathbf{n}\cdot\Delta\boldsymbol{\Psi}}{\|\boldsymbol{\Psi}\|}\,[\mathbf{n}\otimes\mathbf{n}] \\
&+ \left(\left(\frac{\sin(\|\boldsymbol{\Psi}\|/2)}{(\|\boldsymbol{\Psi}\|/2)}\right)^2 - \frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\right)\frac{\mathbf{n}\cdot\Delta\boldsymbol{\Psi}}{\|\boldsymbol{\Psi}\|}\,\widetilde{\boldsymbol{\Psi}} - \frac{1}{2}\left(\frac{\sin(\|\boldsymbol{\Psi}\|/2)}{(\|\boldsymbol{\Psi}\|/2)}\right)^2 \Delta\widetilde{\boldsymbol{\Psi}}
\end{aligned} \tag{6.13}$$

Computation in the limit, when $\|\boldsymbol{\Psi}\| \to 0$, gives:

$$D\mathbf{T}\cdot\Delta\boldsymbol{\Psi}\Big|_{\boldsymbol{\Psi}=0} = -\frac{1}{2}\Delta\widetilde{\boldsymbol{\Psi}} \tag{6.14}$$

The term $\dot{\mathbf{T}}(\mathbf{\Psi})\dot{\mathbf{\Psi}}$ is then computed as follows

$$\begin{aligned}\dot{\mathbf{T}}(\mathbf{\Psi})\dot{\mathbf{\Psi}} &= \left(1+\cos\|\mathbf{\Psi}\| - 2\frac{\sin\|\mathbf{\Psi}\|}{\|\mathbf{\Psi}\|}\right)\frac{\mathbf{n}\cdot\dot{\mathbf{\Psi}}}{\|\mathbf{\Psi}\|}\,\dot{\mathbf{\Psi}} \\ &+ \left[\left(3\,\frac{\sin\|\mathbf{\Psi}\|}{\|\mathbf{\Psi}\|} - \cos\|\mathbf{\Psi}\| - 2\right)\left(\frac{\mathbf{n}\cdot\dot{\mathbf{\Psi}}}{\|\mathbf{\Psi}\|}\right)^2 + \left(1-\frac{\sin\|\mathbf{\Psi}\|}{\|\mathbf{\Psi}\|}\right)\left(\frac{\|\dot{\mathbf{\Psi}}\|}{\|\mathbf{\Psi}\|}\right)^2\right]\mathbf{\Psi} \\ &+ \left(\left(\frac{\sin(\|\mathbf{\Psi}\|/2)}{(\|\mathbf{\Psi}\|/2)}\right)^2 - \frac{\sin\|\mathbf{\Psi}\|}{\|\mathbf{\Psi}\|}\right)\frac{\mathbf{n}\cdot\dot{\mathbf{\Psi}}}{\|\mathbf{\Psi}\|}\,\mathbf{\Psi}\times\dot{\mathbf{\Psi}}\end{aligned} \tag{6.15}$$

At $\mathbf{\Psi}=\mathbf{0}$, this term becomes zero.

The computations for the Euler parameters are more simple, since the matrix $\mathbf{T}(E_0,\mathbf{E})$ is linear:

$$D\mathbf{T}(E_0,\mathbf{E})\cdot\left\{\begin{matrix}\Delta E_0\\ \Delta\mathbf{E}\end{matrix}\right\} = 2\left[-\Delta\mathbf{E}\quad \Delta E_0\mathbf{1}+\widetilde{\Delta\mathbf{E}}\right] \tag{6.16}$$

Then, we see that the correction term for the accelerations becomes:

$$\dot{\mathbf{T}}(E_0,\mathbf{E})\left\{\begin{matrix}\dot{E}_0\\ \dot{\mathbf{E}}\end{matrix}\right\} = 2\left[-\dot{\mathbf{E}}\quad \dot{E}_0\mathbf{1}+\tilde{\dot{\mathbf{E}}}\right]\left\{\begin{matrix}\dot{E}_0\\ \dot{\mathbf{E}}\end{matrix}\right\} = 0 \tag{6.17}$$

A similar computation leads to the following correction term when using the CRV parameters:

$$\begin{aligned}\dot{\mathbf{T}}(\mathbf{C})\dot{\mathbf{C}} &= -\frac{1}{(4-C_0)^3}\left[2\,(\mathbf{C}^T\dot{\mathbf{C}})\,\dot{\mathbf{C}} + \left((4-C_0)\|\dot{\mathbf{C}}\|^2+\frac{1}{2}\,(\mathbf{C}^T\dot{\mathbf{C}})^2\right)\,\mathbf{C}\right. \\ &\left.+\ \frac{\mathbf{C}^T\dot{\mathbf{C}}}{2}\mathbf{C}\times\dot{\mathbf{C}}\right]\end{aligned} \tag{6.18}$$

6.4 Linearization of angular velocities and accelerations

Linearized spatial angular velocities are obtained by computing the Fréchet derivative $D\widetilde{\omega}\cdot\widetilde{\boldsymbol{\theta}}$. To this end, we first calculate the partial results:

$$\begin{aligned}D\dot{\mathbf{R}}\cdot\widetilde{\boldsymbol{\theta}} &= \dot{\widetilde{\boldsymbol{\theta}}}\mathbf{R}+\widetilde{\boldsymbol{\theta}}\dot{\mathbf{R}} \\ D\mathbf{R}^T\cdot\widetilde{\boldsymbol{\theta}} &= -\mathbf{R}^T\widetilde{\boldsymbol{\theta}}\end{aligned} \tag{6.19}$$

where a superposed dot means differentiation with respect to time. Using the latter equations, we are able to express:

$$D\widetilde{\omega}\cdot\widetilde{\boldsymbol{\theta}} = D(\dot{\mathbf{R}}\mathbf{R}^T)\cdot\widetilde{\boldsymbol{\theta}} = \dot{\widetilde{\boldsymbol{\theta}}}+[\,\widetilde{\boldsymbol{\theta}}\widetilde{\omega}-\widetilde{\omega}\widetilde{\boldsymbol{\theta}}\,] \tag{6.20}$$

The term between brackets (Lie bracket) can be expressed in terms of axial vectors by using the identity:

$$[\,\widetilde{\theta}\widetilde{\omega} - \widetilde{\omega}\widetilde{\theta}\,]\,\mathbf{h} = (\boldsymbol{\theta} \times \boldsymbol{\omega}) \times \mathbf{h} \qquad \forall\, \mathbf{h} \in \Re^3 \tag{6.21}$$

Then, equation (6.20) is finally expressed in terms of axial vectors as follows:

$$D\boldsymbol{\omega} \cdot \boldsymbol{\theta} = \dot{\boldsymbol{\theta}} - \boldsymbol{\omega} \times \boldsymbol{\theta} \tag{6.22}$$

By following a similar procedure we obtain:

$$\begin{aligned} D\boldsymbol{\Omega} \cdot \boldsymbol{\theta} &= \mathbf{R}^T \dot{\boldsymbol{\theta}} \\ D\boldsymbol{\omega} \cdot \boldsymbol{\Theta} &= \mathbf{R}\, \dot{\boldsymbol{\Theta}} \\ D\boldsymbol{\Omega} \cdot \boldsymbol{\Theta} &= \dot{\boldsymbol{\Theta}} + \boldsymbol{\Omega} \times \boldsymbol{\Theta} \end{aligned} \tag{6.23}$$

We can correspondingly define the variation of angular velocities:

$$\begin{aligned} \delta\boldsymbol{\omega} &= \delta\dot{\boldsymbol{\theta}} - \boldsymbol{\omega} \times \delta\boldsymbol{\theta} = \mathbf{R}\, \delta\dot{\boldsymbol{\Theta}} \\ \delta\boldsymbol{\Omega} &= \mathbf{R}^T \delta\dot{\boldsymbol{\theta}} = \delta\dot{\boldsymbol{\Theta}} + \boldsymbol{\Omega} \times \delta\boldsymbol{\Theta} \end{aligned} \tag{6.24}$$

The skew-symmetric tensors of spatial and material angular accelerations are defined by time-differentiating the angular velocities (eqs.(6.2)). Linearized angular accelerations are computed by following an entirely similar procedure to that of angular velocities, giving:

$$\begin{aligned} D\boldsymbol{\alpha} \cdot \boldsymbol{\theta} &= \ddot{\boldsymbol{\theta}} - \boldsymbol{\omega} \times \dot{\boldsymbol{\theta}} - \boldsymbol{\alpha} \times \boldsymbol{\theta} \\ D\mathbf{A} \cdot \boldsymbol{\theta} &= \mathbf{R}^T(\ddot{\boldsymbol{\theta}} - \boldsymbol{\omega} \times \dot{\boldsymbol{\theta}}) = \mathbf{R}^T\ddot{\boldsymbol{\theta}} - \boldsymbol{\Omega} \times \mathbf{R}^T\dot{\boldsymbol{\theta}} \\ D\boldsymbol{\alpha} \cdot \boldsymbol{\Theta} &= \mathbf{R}(\ddot{\boldsymbol{\Theta}} + \boldsymbol{\Omega} \times \dot{\boldsymbol{\Theta}}) = \mathbf{R}\ddot{\boldsymbol{\Theta}} + \boldsymbol{\omega} \times \mathbf{R}\dot{\boldsymbol{\Theta}} \\ D\mathbf{A} \cdot \boldsymbol{\Theta} &= \ddot{\boldsymbol{\Theta}} + \boldsymbol{\Omega} \times \dot{\boldsymbol{\Theta}} + \mathbf{A} \times \boldsymbol{\Theta} \end{aligned} \tag{6.25}$$

We remark that only variations with respect to angular displacements in the spatially fixed inertial frame will be employed afterwards to compute the inertia forces.

The developed expressions express variations with respect to increments that lie in the tangent space to $SO(3)$ at the actual rotation. They are independent of the technique employed to parametrize the operator $\mathbf{R}$. By projecting the increments variations to the chosen parameter space, we will be able to express the variations of the physical magnitudes in terms of variations of the rotation parameters.

6.5 Angular velocities and accelerations in a moving frame

In different applications, we will need to compute the velocities and accelerations in an inertial frame from measures taken in a moving frame. For instance, in the superelement formulation we will express that the actual rotation at one point of the structure results from the composition of the rotation of a frame attached to the considered point relative to a reference frame, superposed onto the rotation of the reference frame itself:

$$\mathbf{R} = \mathbf{R}_0 \, \mathbf{R}_{rel} \tag{6.26}$$

We remark that the reference rotation $\mathbf{R}_0$ is not fixed, but instead it gives the orientation of a frame that follows the mean motion of the body. Then, in our computations we should consider terms proceeding from the variations of $\mathbf{R}_0$.

Let us for instance compute the variation of angular displacements. By differentiating both sides of equation (6.26), we get

$$\mathbf{R} \, \delta\widetilde{\Theta} = \mathbf{R}_0 \, \delta\widetilde{\Theta}_0 \mathbf{R}_{rel} + \mathbf{R}_0 \mathbf{R}_{rel} \delta\widetilde{\Theta}_{rel} \tag{6.27}$$

Now we premultiply by $\mathbf{R}^T$ to obtain:

$$\delta\widetilde{\Theta} = \mathbf{R}_{rel}^T \, \delta\widetilde{\Theta}_0 \, \mathbf{R}_{rel} + \delta\widetilde{\Theta}_{rel} \tag{6.28}$$

This equation is rewritten in terms of axial vectors as follows

$$\delta\Theta = \mathbf{R}_{rel}^T \, \delta\Theta_0 + \delta\Theta_{rel} \quad , \tag{6.29}$$

equation from which we finally obtain, in terms of rotational vectors:

$$\mathbf{T}(\Psi) \, \delta\Psi = \mathbf{R}_{rel}^T \, \mathbf{T}(\Psi_0) \, \delta\Psi_0 + \mathbf{T}(\Psi_{rel}) \delta\Psi_{rel} \tag{6.30}$$

The same procedure can be followed to compute the angular velocity in terms of the velocity of the reference frame and of the relative velocity

$$\Omega = \mathbf{R}_{rel}^T \, \Omega_0 + \Omega_{rel} \tag{6.31}$$

After time differentiating the latter equation we obtain the expression of the angular accelerations:

$$\mathbf{A} = \mathbf{R}_{rel}^T \, \mathbf{A}_0 + \mathbf{A}_{rel} - \Omega_{rel} \times \mathbf{R}_{rel}^T \Omega_0 \tag{6.32}$$

Similar expressions could be developed for the other parametrization techniques.

7. INCREMENTAL ROTATIONS AS UNKNOWNS

Nonlinear structural problems, either in statics or in dynamics, are formulated in a sequential form: the final solution is obtained by solving a sequence of partial problems. For instance, step-by-step algorithms are used to time integrate the nonlinear ordinary differential equations that constitute the nonlinear dynamic problem. Then, we are able to express the problem to be solved in an incremental way: we know a previous solution and we want to determine the increment necessary to obtain the new admissible configuration.

Then, rotations are to be treated in an incremental form. At each stage of the solution process one determines the incremental rotation necessary to carry from the previous converged configuration (or reference configuration), to the current one:

$$\mathbf{R} = \mathbf{R}_{ref}\,\mathbf{R}_{inc} \tag{7.1}$$

This equation is rewritten in terms of rotational vectors as follows:

$$\exp(\widetilde{\mathbf{\Psi}}) = \exp(\widetilde{\mathbf{\Psi}}_{ref})\;\exp(\widetilde{\mathbf{\Psi}}_{inc}) \tag{7.2}$$

where $\mathbf{\Psi}$ is the rotational vector corresponding to the actual rotation $\mathbf{R}$, $\mathbf{\Psi}_{ref}$ is the rotational vector corresponding to the reference rotation $\mathbf{R}_{ref}$, and $\mathbf{\Psi}_{inc}$ is the rotational vector corresponding to the incremental rotation $\mathbf{R}_{inc}$.

This approach can be seen as an updated Lagrangian point of view. The reference rotation is fixed, so that the expressions of variation of angular displacements, velocities and accelerations are simply obtained in terms of the rotational vector of the incremental rotation by replacing $\mathbf{\Psi}$ by $\mathbf{\Psi}_{inc}$ into eqns. (5.36, 6.6, 6.10) :

$$\begin{aligned}
\delta\Theta &= \mathbf{T}(\mathbf{\Psi}_{inc})\;\delta\mathbf{\Psi}_{inc} \\
\Omega &= \mathbf{T}(\mathbf{\Psi}_{inc})\;\dot{\mathbf{\Psi}}_{inc} \\
\mathbf{A} &= \mathbf{T}(\mathbf{\Psi}_{inc})\;\ddot{\mathbf{\Psi}}_{inc} + \dot{\mathbf{T}}(\mathbf{\Psi}_{inc})\;\dot{\mathbf{\Psi}}_{inc}
\end{aligned} \tag{7.3}$$

8. CONCLUDING REMARKS

The representation of finite rotations has been studied, and the different tools available to describe them were presented. The rotation operator was derived starting from different approaches and the problem of parametrizing it was given a particular attention.

Various systems of parametrizations were discussed, i.e.:

- Euler angles
- Bryant angles
- Euler parameters
- Rodrigues parameters
- The conformal rotation vector (CRV)
- The rotational vector
- Linear parameters

In the following, we will make a brief comparative study of these systems:

i) A first distinction can be made from the point of view of invariance: Euler and Bryant angles are non-invariant measures: they are oriented to treating particular problems and so, they are not well suited to describing general problems. Also, they present a point of singularity. The other parametrizations rely on the Euler representation of a finite rotation by means of an axis and an angle of rotation.

ii) Euler parameters, CRV parameters and the rotational vector can be used to represent rotations of any magnitude in the interval $\phi \in [-\pi, \pi]$. On the other side, Rodrigues and linear parameters present a point of singularity at $|\phi| = \pi$.

With regard to rotations out of the interval $[-\pi, \pi]$, Euler parameters and the rotational vector do not present singularities, while the CRV's go to infinity when $|\phi| \to 2\pi$.

iii) When analyzing the differentiability properties of the rotational vector and Euler and CRV parameters, we see that all of them have a continuous and invertible derivative $\mathbf{T}$ in $\phi \in [-\pi, \pi]$. However, the only parametrization that allows going to any magnitude of rotation (out of the interval $[-\pi, \pi]$) is the Euler parameters system. The rotational vector presents a singular point ($\mathbf{T}$ is not invertible) at $|\phi| = 2\pi$ and the CRV's parameters at $|\phi| = 3.709$.

The angle limitation can be easily surmounted for the rotational vector and CRV parameters, by working only in the interval $|\phi| \in [-\pi, \pi]$ and transforming rotations that exceed this range to rotations within it by simply adding or subtracting 2π to the rotation angle. However, we should note that at $|\phi| = \pi$, the CRV's parameters are closer to the singular point than the rotational vector is, so one would expect a better behavior of the latter parametrization from the point of view of convergence of the numerical algorithm.

If we limit now the analysis to those parametrizations capable of handling any magnitude of rotation, we can say in addition:

iv) The rotational vector and the CRV's parameters have the advantage of working with a minimal set of parameters, while the Euler's form a set of four dependent parameters linked by a constraint. This poses the inconvenience of requiring the addition of a Lagrange multiplier to account for the constraint at each (rotational) node of the discretization.

v) Euler parameters lead to simple equations, while CRV's and the rotational vector give more complex ones, specially for the angular accelerations evaluation.

vi) CRV's and Euler parameters do not require the evaluation of trigonometric functions, while the rotational vector implies the computation of two trigonometric functions to obtain the rotation operator.

vii) A simple composition rule can be derived from quaternion theory for the Euler parameters representation. It can be also extended to the CRV's parameters. The rotational vector can not give a simple composition rule based on this theory since it does not form a quaternion. Another parameters system that verifies the quaternion composition rule is the Rodrigues parameters system.

viii) The rotational vector gives a more simple geometric characterization of the rotation, while the others require the interpretation of the result in terms of the values taken by more or less complicated trigonometric functions.

As a final conclusion, concerning the selection of a parametrization system to be applied in a general mechanisms problem, we can say that if one accepts to pay the cost of handling four rotation parameters per node with the consequent constraint equation, the Euler parameters are the better choice since they give fairly simple equations with a small degree of nonlinearity. On the other side, if one wants to employ a three-parameters system both the CRV's set and the rotational vector can be conveniently employed, but we consider the rotational vector could be a better choice since it has a direct geometrical meaning and also it could give better convergence properties of the nonlinear solution algorithms.

References

1. ANGELES J., *"Spatial kinematic chains - Analysis, synthesis and optimization"*, Springer - Verlag, Berlin (1982).

2. MILENKOVIC V., "Coordinates suitable for angular motion synthesis in robots", *Proceedings of ROBOT 6 conference* (1982).

3. GERADIN M. and CARDONA A., "Kinematics and dynamics of rigid and flexible mechanisms using finite elements and quaternion algebra", to appear in *Computational Mechanics* (1987).

4. ARGYRIS J., "An excursion into large rotations", *Comp. Meth. Appl. Mech. Engng.*, Vol.32, pp.85-155 (1982).

5. SPURRIER R.A., "Comment on 'Singularity-free extraction of a quaternion from a direction-cosine matrix' ", *Journal of Spacecraft*, Vol.15, p.255 (1978).

6. WEHAGE R. A., "Quaternions and Euler parameters. A brief exposition", in *Computer aided analysis and optimization of mechanical system dynamics*, Ed. by E. J. HAUG, Springer-Verlag (1984).

7. SPRING K. W., "Euler parameters and the use of quaternion algebra in the manipulation of finite rotations: a review", *Mechanism and Machine Theory*, Vol.21, pp.365-373 (1986).

8. HAMILTON W.R., *"Elements of quaternions"*, Cambridge University Press, Cambridge (1899).

9. SIMO J.C. and VU-QUOC L., "A three-dimensional finite-strain rod model. Part II: computational aspects", *Comp. Meth. Appl. Mech. Engng.*, Vol.58, pp.79-116 (1986).