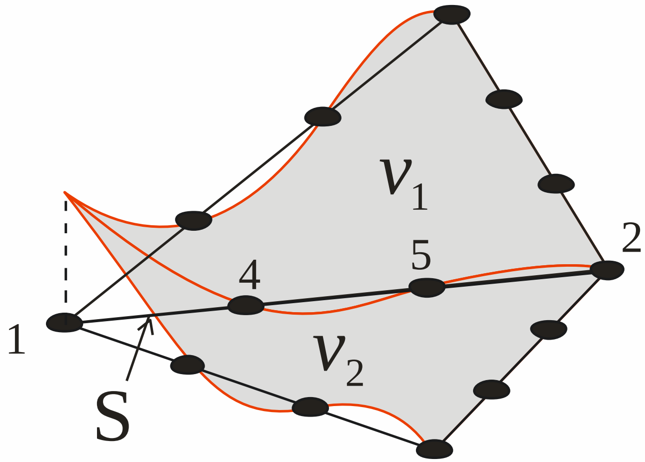

Introducción al Método de los Elementos Finitos

Elementos Mixtos

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Elementos Mixtos

Generalizan los EF vistos p/problemas elípticos.

Ej típico: problema Stokes 2D.

Hallar $\underline{u}=(u_1,u_2)$ y p (presión)

velocidad

campo div nulo

$$\boxed{\begin{array}{ll} -\Delta \underline{u} + \underline{\nabla} p = f & \text{en } \Omega \\ \text{div } \underline{u} = 0 & \text{en } \Omega \\ \underline{u} = 0 & \text{sobre } \Gamma \end{array}}$$

Nota: Si p es solución $\Rightarrow$ $(p+c)$ es soluc $\forall c$ cte

p/lograr p única $\Rightarrow$ $\left| \int_\Omega p \, dx = 0 \right|$

Formulación variacional:

Hallar $u \in V$ y $p \in H$ /

$$V = [H_0^1(\Omega)]^2 = \{ \underline{v} = (v_1, v_2) / v_i \in H_0^1(\Omega), i=1,2 \}$$

$$H = \{ q \in L_2(\Omega) : \int_\Omega q \, dx = 0 \}$$

$$-(v, \Delta u) + (v, \nabla p) = (f, v) \quad \forall v \in V$$

Int p/partes:

Hallar $\underline{u}$, p en V, H /

$$\boxed{\begin{aligned} &(\nabla v, \nabla u) - (\operatorname{div} v, p) = (f, v) \quad && (1) \\ &(q, \operatorname{div} u) = 0 && \forall q \in H \quad (2) \end{aligned}}$$

donde $(\, , \,)$ prod int en L_2.

En particular:

$$(\nabla u, \nabla v) = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i \, d\Omega = \sum \int \frac{\partial u_i}{\partial x_1}\frac{\partial v_i}{\partial x_1} + \frac{\partial u_i}{\partial x_2}\frac{\partial v_i}{\partial x_2}$$

$$\Delta \underline{u} - \underline{\nabla} p \equiv \begin{cases} \dfrac{\partial^2 u_1}{\partial x_1^2} + \dfrac{\partial^2 u_1}{\partial x_2^2} - \dfrac{\partial p}{\partial x_1} \\[2ex] \dfrac{\partial^2 u_2}{\partial x_1^2} + \dfrac{\partial^2 u_2}{\partial x_2^2} - \dfrac{\partial p}{\partial x_2} \end{cases}$$

$$(v, \Delta u) = \int v_1 \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) + v_2 \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2}\right) d\Omega$$

$$(f, v) = \int_\Omega f_i v_i \, dx$$

Prob variacional discreto

$V \rightarrow V_h \qquad H \rightarrow H_h$

Hallar $(u_h, p_h) \in V_h \times H_h$ /

$$(*)\quad \begin{cases} (\nabla u_h, \nabla v) - (p_h, \operatorname{div} v) = (f, v) & \forall v \in V_h \\ (q, \operatorname{div} u_h) = 0 & \forall q \in H_h \end{cases}$$

Un método q/resuelve es un EF "Mixto"

Nota: la condición de divergencia nula se cumple en forma aproximada.

Costo adicional $\Longrightarrow$ espacio H_h.

Ejemplo “bloqueo”

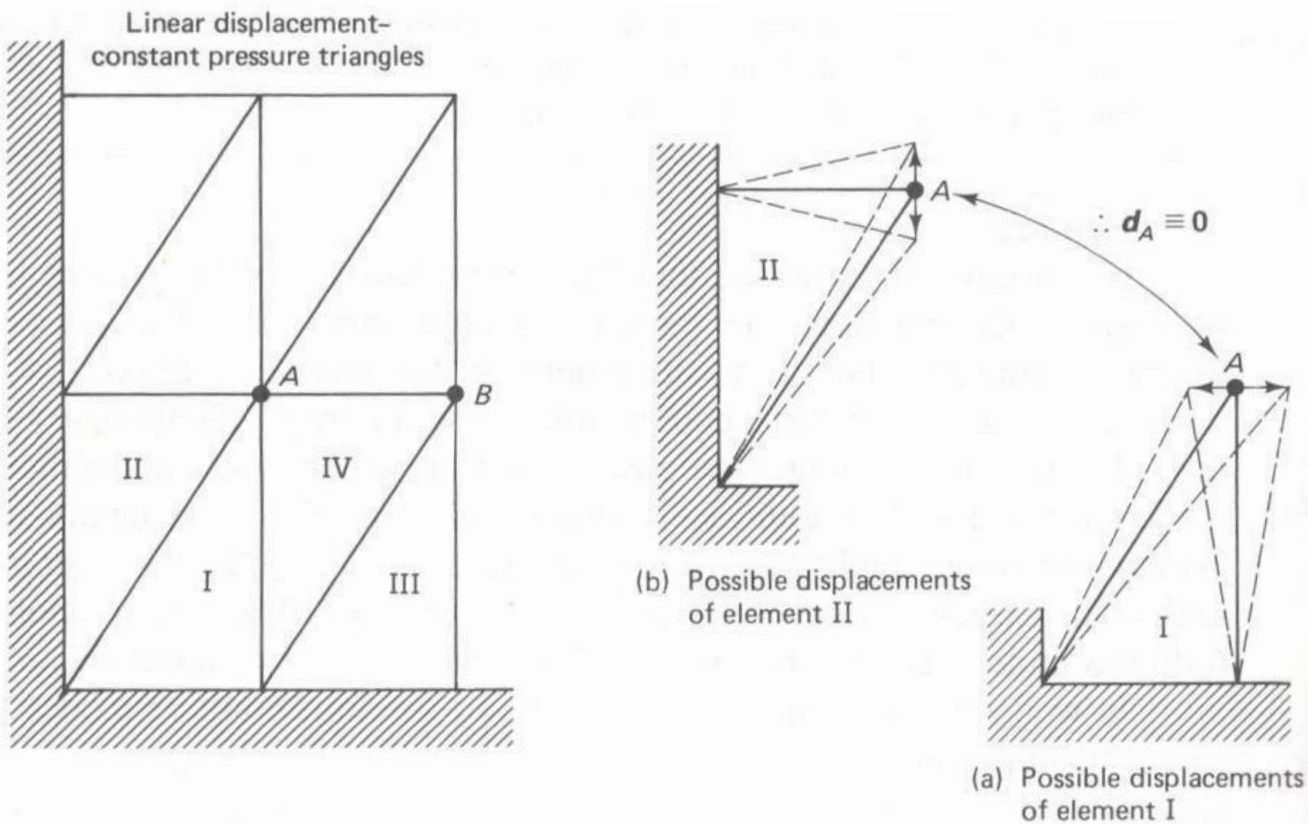
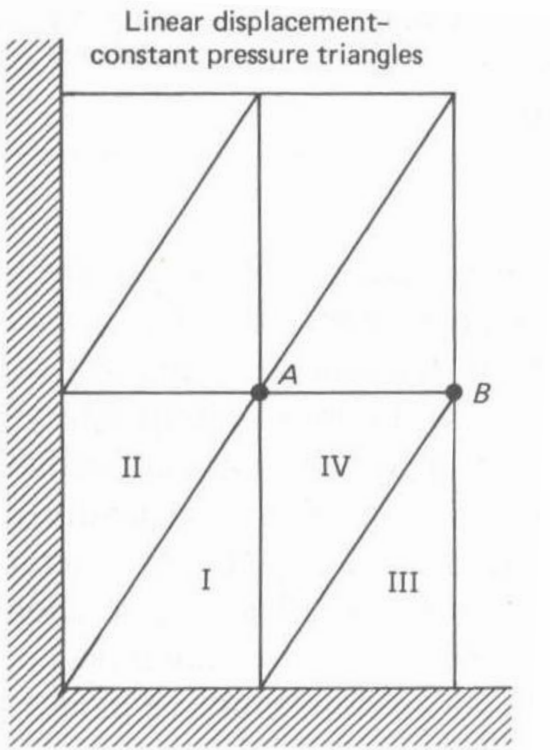


Figure 4.3.2 Mesh for which incompressibility dictates zero displacements.

Ejemplo "bloqueo"

$$\left(q, \mathrm{div}\,\mathbf{u}_h\right) = 0 \qquad \forall\, q \in H_h \quad \Longrightarrow \quad \begin{aligned} &\left(q_{\mathrm{I}}, \mathrm{div}\,\mathbf{u}_h\right) = 0 \\ &\left(q_{\mathrm{II}}, \mathrm{div}\,\mathbf{u}_h\right) = 0 \\ &\ldots \\ &\left(q_{\mathrm{VIII}}, \mathrm{div}\,\mathbf{u}_h\right) = 0 \end{aligned}$$



$$q_i\left(\mathbf{x}\right) = \begin{cases} 1 & \mathbf{x} \in K_i \\ 0 & \text{fuera} \end{cases}$$

Luego

$$\left(q_i, \mathrm{div}\,\mathbf{u}_h\right) = \int_\Omega q_i \mathrm{div}\,\mathbf{u}_h \, d\Omega = \int_{K_i} \mathrm{div}\,\mathbf{u}_h \, d\Omega = 0$$

Pero

$$\mathbf{u}_h\left(\mathbf{x}\right)\Big|_{K_I} = \frac{y}{h_y}\begin{Bmatrix} U_A \\ V_A \end{Bmatrix} \quad \Longrightarrow \quad \mathrm{div}\,\mathbf{u}_h\left(\mathbf{x}\right)\Big|_{K_I} = \frac{V_A}{h_y}$$

$$\int_{K_I} \mathrm{div}\,\mathbf{u}_h \, d\Omega = \frac{\Omega_{K_I}}{h_y} V_A = 0$$

De manera similar

$$\int_{K_{II}} \mathrm{div}\,\mathbf{u}_h \, d\Omega = \frac{\Omega_{K_{II}}}{h_x} U_A = 0$$

Veremos la elección de V_h y H_h no es totalmente libre. No toda combinación funciona.

Veremos 3 casos de V_h, H_h / estabilidad se prueba fácilmente pero no resulta un método óptimamente preciso.

Estabilidad:

$\exists$ una constante C / si $(u_h, p_h) \in V_h \times H_h$ satisface $(*)$ luego:

$(**)$ $$\|u_h\|_1 + \|p_h\|_0 \leq C \|f\|_{-1}$$

$$\|f\|_{-1} = \sup_{v \in V} \frac{(f,v)}{\|v\|_1} \qquad \|v\|_1^2 = \|v_1\|_{H^1}^2 + \|v_2\|_{H^1(\Omega)}^2$$

$$\|q\|_0 = \|q\|_{L_2(\Omega)}$$

La estimación (**) p/ la velocidad se obtiene fácilmente:

$$(\nabla u_h, \nabla u_h) - (p_h, \operatorname{div} u_h) = (f, u_h)$$

$$+ \quad (p_h, \operatorname{div} u_h) = 0$$

$$\|\nabla u_h\|_{L_2}^2 = (\nabla u_h, \nabla u_h) = (f, u_h) \leq \|f\|_{-1} \|u_h\|_1$$

Hay que llegar a:

$$\boxed{\|u_h\|_1 \leq C \|f\|_{-1}}$$

(*) $(p_h, \operatorname{div} v) = (\nabla u_h, \nabla v) - (f, v) \quad \forall v \in V_h$

Queremos partir de ésta y llegar a:

(*) $\|p_h\|_0 \leq C\left(\|u_h\|_1 + \|f\|_{-1}\right)$

P/lograr (*) a partir de (*) se necesita lo sig:

$\exists$ una cte $c > 0$ / $\forall\, q \in H_h$

$$\sup_{v \in V_h} \frac{(q, \operatorname{div} v)}{\|v\|_1} \geq c\,\|q\|_0$$

Condic de Babuska Brezzi

Babuska - Brezzi

$$(p_h, \operatorname{div} v) = (\nabla u_h, \nabla v) - (f, v) \quad \forall v \in V_h$$

$$\|u_h\|_1 \le C \|f\|_{-1}$$

$$|(p_h, \operatorname{div} u_h)| \le |(\nabla u_h, \nabla u_h)| + |(f, u_h)| \le$$

?

$$\le \|u_h\|_1^2 + \|f\|_{-1} \|u_h\|_1$$

NO

$$c\|p_h\|_0 \le \frac{|(p_h, \operatorname{div} u_h)|}{\|u_h\|_1} \le \|u_h\|_1 + \|f\|_{-1}$$

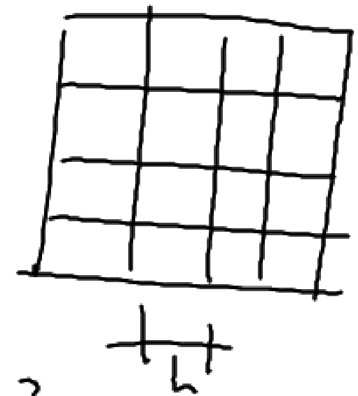
$$\sup_{v \in V_h} \frac{(q, \operatorname{div} v)}{\|v\|_1} \ge c \|q\|_0$$

Ejemplo

Asumimos Ω cuadrado

$$T_h = \{K\}$$

K cuadrados de lado h



$$V_h = \{ v \in V : v|_K \in [Q_2(K)]^2 \ \forall K \in T_h \}$$

$$H_h = \{ q \in H : q|_K \in Q_0(K) \ \forall K \in T_h \}$$

Veremos si cumple BB

Si es así cumple la estimación de error:

$$\|u-u_h\|_1 + \|p-p_h\|_0 \leq Ch\left(h\|u\|_3 + \|p\|_1\right)$$

Resultado preliminar (Girault-Raviart)

Se demuestra

existe una constante C / $\forall q \in H$ $\exists\, v \in [H_0^1(\Omega)]^2$
que verifica

$$\boxed{\begin{aligned} \operatorname{div} v &= q \\ \|v\|_1 &\leq C \|q\|_0 \end{aligned}} \Leftarrow (***)$$

Dada q $\Rightarrow$ sea v / $\operatorname{div} v = q$

Luego

$$\|v\|_1 \leq C \|q\|_0$$

Tiene relación c/ BB:

$$\frac{(q, \operatorname{div} v)}{\|v\|_1} \geq \frac{(q, q)}{\|v\|_1} = \frac{\|q\|_0^2}{C \cancel{\|q\|_0}}$$

Luego $\sup_{v_1 \in V} \frac{(q, \operatorname{div} v_1)}{\|v_1\|_1} \geq c \|q\|_0 \quad \forall q \in H$

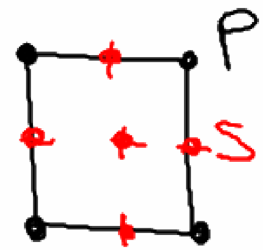
BB P/ Stokes continuo

Vemos q/ para P/ variacional discreto.

Sea $q \in H_h$. Sea $v \in V$ / $\operatorname{div} v = q$

Sea $v_h \in V_h$ un interpolante de v:

$$\begin{cases} v_h(P) = \tilde{v}(P) \\ \int_S v_h \, ds = \int_S v \, ds \quad \forall \text{ lados } S \\ \int_K v_h \, dx = \int_K v \, dx \quad \forall K \in T_h \end{cases}$$



donde $\tilde{v} \in V_h$ / $(\nabla(v - \tilde{v}), \nabla w) = 0 \quad \forall w \in V_h$

1) Es fácil ver $\|v_h\|_1 \leq C\|v\|_1$ Ejercicio

2)
$$\|q\|_0^2 = (q, \operatorname{div} v) = \sum_K \int_K q \operatorname{div} v \, dx \underset{\text{Green}}{=}$$

$$= \sum_K \int_{\partial K} q \, \underline{v} \cdot \underline{n}_K \, ds \underset{q \text{ const/elemento}}{=} \sum_K \int_{\partial K} q \, v_h \cdot n_K \, ds$$

$$= \sum_K \int_K q \operatorname{div} v_h \, d\underline{x} = (q, \operatorname{div} v_h)$$

Por (***)

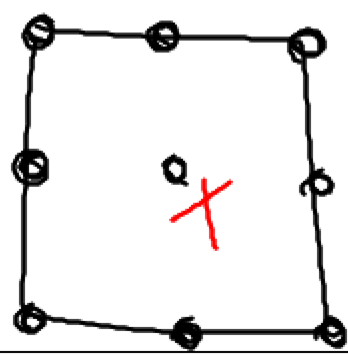
Por (1)

$$\|v_h\|_1 \leq C\|v\|_1 \leq C\|q\|_0$$

$$\|q\|_0 = \frac{(q, \operatorname{div} v_h)}{\|q\|_0} \le C \, \frac{(q, \operatorname{div} v_h)}{\|v_h\|_1}$$

BB

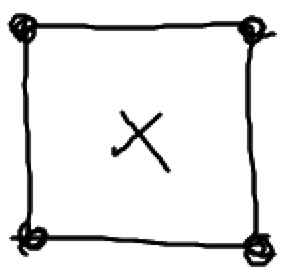
$$\sup_{v \in V_h} \frac{(q, \operatorname{div} v)}{\|v\|_1} \ge C \, \|q\|_0$$



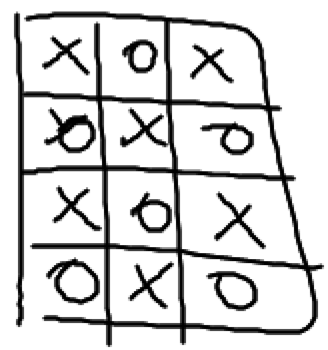
Ej 2) El elemento + simple sería: $Q_1 - Q_0$

$$V_h = \{ v \in V : v|_K \in [Q_1(K)]^2 \ \forall K \in T_h \}$$

$$H_h = \{ q \in H : q|_K \in Q_0(K) \ \forall K \in T_h \}$$



No satisface BB

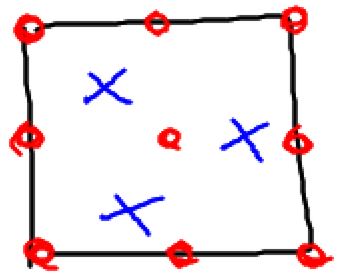


Hay soluciones procedimiento p/ filtrar modos "espúreos"

Ej 3

$$V_h = \left\{ v \in V : v|_K \in [Q_2(K)]^2 \right\}$$

$$H_h = \left\{ q \in H : q|_K \in P_1(K) \right\}$$



$Q_2 - P_1$

Ej 4

$Q_2 - Q_1 \Rightarrow$ inestable

Problema de Stokes – Formulación mixta

- Consideremos nuevamente las ecuaciones de Stokes para el flujo estacionario de un fluido Newtoniano incompresible encerrado en un dominio $\Omega \subset \mathbb{R}^2$, sometido a una fuerza volumétrica $\boldsymbol{f}$:

$\sigma_{ij,j} + f_i = 0$	en Ω,	Balance de cant. de movto.	$\boldsymbol{u}$: velocidad
$\sigma_{ij} = 2\mu\varepsilon_{ij}(\boldsymbol{u}) - p\delta_{ij}$	en Ω,	Ley const. de fluido Newtoniano	$\boldsymbol{\sigma}$: tensión
$u_{i,i} = 0$	en Ω,	Condición de incompresibilidad	p: presión
$u_i = 0$	sobre Γ,	CB Dirichlet	μ: viscosidad

$-\mu\Delta u_i + p_{,i} = f_i$ en Ω, Balance de cant. de movto. p/fluido Newtoniano

- Definimos los espacios de funciones de prueba

$$V = \left\{ \boldsymbol{v} : \boldsymbol{v} \in \left[H_0^1(\Omega) \right]^2 \text{ en } \Omega \right\} = \left\{ \boldsymbol{v} = (v_1, v_2) / v_i \in H_0^1(\Omega), \quad i = 1, 2 \right\}$$

$$H = \left\{ q \in L_2(\Omega) : \int_\Omega q \, d\boldsymbol{x} = 0 \right\}$$

Forma variacional del problema de Stokes

- Luego, podemos llevar el problema de Stokes a la forma variacional

$$(V) \qquad \text{Hallar } \boldsymbol{u} \in \mathrm{V} \text{ y } p \in \mathrm{H} \quad /$$

$$\begin{cases} \mu\left(\nabla \boldsymbol{v}, \nabla \boldsymbol{u}\right) - \left(\operatorname{div} \boldsymbol{v}, p\right) = (f, \boldsymbol{v}), & \forall \boldsymbol{v} \in \mathrm{V} \\ \left(q, \operatorname{div} \boldsymbol{u}\right) = 0, & \forall q \in \mathrm{H} \end{cases}$$

donde $(\ ,\ )$ es el producto interno en L_2 .

- Para llevar el problema de Stokes a la forma variacional hacemos

$$f_i = -\mu \Delta u_i + p_{,i}$$

$$\int_\Omega f_i v_i dx = -\mu \int_\Omega \Delta u_i v_i dx + \int_\Omega p_{,i} v_i dx$$

$$\int_\Omega f_i v_i dx = \mu \int_\Omega \nabla u_i \cdot \nabla v_i \, dx - \int_\Omega p v_{i,i} dx - \underbrace{\mu \int_\Gamma \frac{\partial u_i}{\partial n} v_i ds}_{=0} + \underbrace{\int_\Gamma p n_i v_i ds}_{=0}$$

$$u_{i,i} = 0$$

$$\int_\Omega q u_{i,i} dx = 0$$

Elemento Q1P0

$$\boldsymbol{u} = \sum_{j=1}^{4} \varphi_{\mathrm{j}}(\boldsymbol{x})\ \boldsymbol{U}^{j}, \qquad \boldsymbol{x} \in \mathrm{K}$$

$$\boldsymbol{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 & \varphi_4 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 & \varphi_4 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \\ u_1^4 \\ u_2^4 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \end{bmatrix} \boldsymbol{U} = \boldsymbol{\varphi} \boldsymbol{U}$$

$$\nabla v_i \bullet \nabla u_i = \nabla v_1 \bullet \nabla u_1 + \nabla v_2 \bullet \nabla u_2$$

$$\nabla u_1 = \sum_{j=1}^{4} \nabla \varphi_{\mathrm{j}}(\boldsymbol{x})\ u_1^j = \nabla \boldsymbol{\varphi}_1 \boldsymbol{U}, \qquad \nabla u_2 = \sum_{j=1}^{4} \nabla \varphi_{\mathrm{j}}(\boldsymbol{x})\ u_2^j = \nabla \boldsymbol{\varphi}_2 \boldsymbol{U}$$

$$\nabla \boldsymbol{\varphi}_1 = \begin{bmatrix} \nabla \varphi_1 & 0 & \nabla \varphi_2 & 0 & \nabla \varphi_3 & 0 & \nabla \varphi_4 & 0 \end{bmatrix}$$

$$\nabla \boldsymbol{\varphi}_2 = \begin{bmatrix} 0 & \nabla \varphi_1 & 0 & \nabla \varphi_2 & 0 & \nabla \varphi_3 & 0 & \nabla \varphi_4 \end{bmatrix}$$

$$\boldsymbol{A}^{\mathrm{K\,VV}} = \mu \int_{\mathrm{K}} \left(\nabla \boldsymbol{\varphi}_1{}^T \nabla \boldsymbol{\varphi}_1 + \nabla \boldsymbol{\varphi}_2{}^T \nabla \boldsymbol{\varphi}_2 \right) d\boldsymbol{x}$$

Elemento Q1P0

$$\operatorname{div}\boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \sum_{j=1}^{4} \frac{\partial \varphi_j}{\partial x_1}\ \boldsymbol{U}^j + \sum_{j=1}^{4} \frac{\partial \varphi_j}{\partial x_2}\ \boldsymbol{U}^j$$

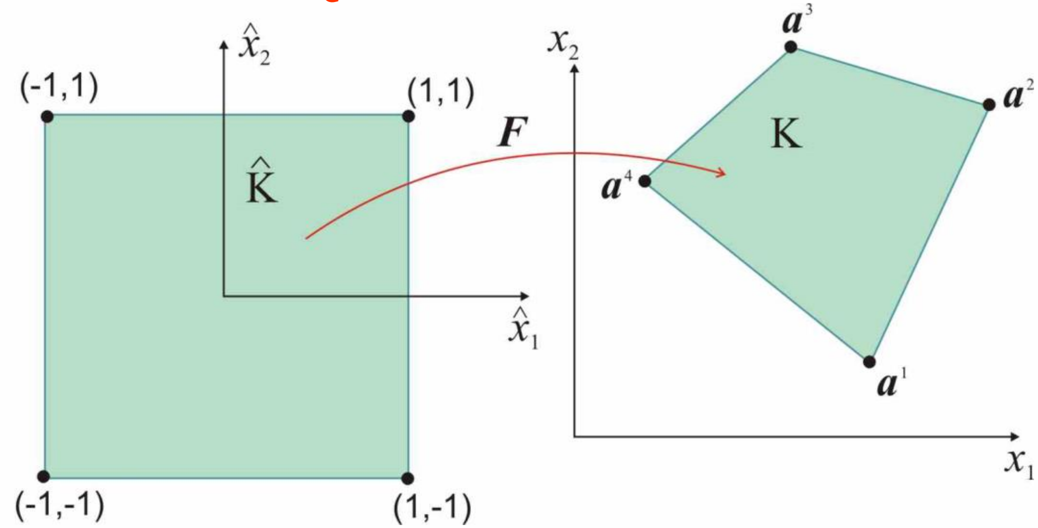
$$= \left[\frac{\partial \varphi_1}{\partial x_1} \quad \frac{\partial \varphi_1}{\partial x_2} \quad \frac{\partial \varphi_2}{\partial x_1} \quad \frac{\partial \varphi_2}{\partial x_2} \quad \frac{\partial \varphi_3}{\partial x_1} \quad \frac{\partial \varphi_3}{\partial x_2} \quad \frac{\partial \varphi_4}{\partial x_1} \quad \frac{\partial \varphi_4}{\partial x_2} \right] \boldsymbol{U} = \boldsymbol{\psi}\, \boldsymbol{U}$$

$$p = 1\, \boldsymbol{P}, \qquad \boldsymbol{x} \in \mathrm{K} \text{ (cte en el elemento)}$$

$$\boldsymbol{A}^{\mathrm{K\ VP}} = -\int_{\mathrm{K}} \boldsymbol{\psi}^T\ d\boldsymbol{x}$$

$$\left(\boldsymbol{A}^{\mathrm{K}}\right)_{9x9} = \begin{bmatrix} \boldsymbol{A}^{\mathrm{K\ VV}} & \boldsymbol{A}^{\mathrm{K\ VP}} \\ \boldsymbol{A}^{\mathrm{K\ PV}} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \mu \int_{\mathrm{K}} \left(\nabla \boldsymbol{\varphi}_1{}^T \nabla \boldsymbol{\varphi}_1 + \nabla \boldsymbol{\varphi}_2{}^T \nabla \boldsymbol{\varphi}_2 \right) d\boldsymbol{x} & -\int_{\mathrm{K}} \boldsymbol{\psi}^T\ d\boldsymbol{x} \\ -\int_{\mathrm{K}} \boldsymbol{\psi}\, d\boldsymbol{x} & \boldsymbol{0} \end{bmatrix}$$

Elemento isoparamétrico Q1P0



Funciones de base en elemento máster:

$$\hat{\varphi}_1(\hat{x}_1,\hat{x}_2)=\frac{1}{4}(1+\hat{x}_1)(1+\hat{x}_2)$$

$$\hat{\varphi}_2(\hat{x}_1,\hat{x}_2)=\frac{1}{4}(1-\hat{x}_1)(1+\hat{x}_2)$$

$$\hat{\varphi}_3(\hat{x}_1,\hat{x}_2)=\frac{1}{4}(1-\hat{x}_1)(1-\hat{x}_2)$$

$$\hat{\varphi}_4(\hat{x}_1,\hat{x}_2)=\frac{1}{4}(1+\hat{x}_1)(1-\hat{x}_2)$$

$$\boldsymbol{x}=\boldsymbol{F}(\hat{\boldsymbol{x}})=\sum_{j=1}^{4}\hat{\varphi}_j(\hat{\boldsymbol{x}})\ \boldsymbol{a}^j,\qquad \hat{\boldsymbol{x}}\in\hat{K}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\sum_{j=1}^{4}\hat{\varphi}_j(\hat{\boldsymbol{x}})\ \begin{Bmatrix} a_1^j \\ a_2^j \end{Bmatrix}$$

Elemento isoparamétrico Q1P0

$$\boldsymbol{x} = \boldsymbol{F}(\hat{\boldsymbol{x}}) = \sum_{k=1}^{4} \hat{\varphi}_k(\hat{\boldsymbol{x}})\ \boldsymbol{a}^k, \qquad \hat{\boldsymbol{x}} \in \hat{K}$$

$$\boldsymbol{F}(\hat{\boldsymbol{x}}) = \begin{Bmatrix} F_1(\hat{\boldsymbol{x}}) \\ F_2(\hat{\boldsymbol{x}}) \end{Bmatrix} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \end{bmatrix} \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \\ \hat{\varphi}_3 \\ \hat{\varphi}_4 \end{bmatrix}$$

$$\frac{\partial \boldsymbol{x}}{\partial \hat{x}_j} = \frac{\partial \boldsymbol{F}}{\partial \hat{x}_j}(\hat{\boldsymbol{x}}) = \sum_{k=1}^{4} \frac{\partial \hat{\varphi}_k}{\partial \hat{x}_j}(\hat{\boldsymbol{x}})\ \boldsymbol{a}^k$$

$$\boldsymbol{J}(\hat{\boldsymbol{x}}) = \frac{\partial \boldsymbol{F}}{\partial \hat{\boldsymbol{x}}}(\hat{\boldsymbol{x}}) = \begin{bmatrix} \frac{\partial F_1}{\partial \hat{x}_1} & \frac{\partial F_1}{\partial \hat{x}_2} \\ \frac{\partial F_2}{\partial \hat{x}_1} & \frac{\partial F_2}{\partial \hat{x}_2} \end{bmatrix} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\varphi}_1}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_1}{\partial \hat{x}_2} \\ \frac{\partial \hat{\varphi}_2}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_2}{\partial \hat{x}_2} \\ \frac{\partial \hat{\varphi}_3}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_3}{\partial \hat{x}_2} \\ \frac{\partial \hat{\varphi}_4}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_4}{\partial \hat{x}_2} \end{bmatrix}$$

$$\frac{\partial \hat{\varphi}_1}{\partial \hat{x}_1} = \frac{1}{4}(1+\hat{x}_2) \qquad \frac{\partial \hat{\varphi}_1}{\partial \hat{x}_2} = \frac{1}{4}(1+\hat{x}_1)$$

$$\frac{\partial \hat{\varphi}_2}{\partial \hat{x}_1} = -\frac{1}{4}(1+\hat{x}_2) \qquad \frac{\partial \hat{\varphi}_2}{\partial \hat{x}_2} = \frac{1}{4}(1-\hat{x}_1)$$

$$\frac{\partial \hat{\varphi}_3}{\partial \hat{x}_1} = -\frac{1}{4}(1-\hat{x}_2) \qquad \frac{\partial \hat{\varphi}_3}{\partial \hat{x}_2} = -\frac{1}{4}(1-\hat{x}_1)$$

$$\frac{\partial \hat{\varphi}_4}{\partial \hat{x}_1} = \frac{1}{4}(1-\hat{x}_2) \qquad \frac{\partial \hat{\varphi}_4}{\partial \hat{x}_2} = -\frac{1}{4}(1+\hat{x}_1)$$

$$\nabla \hat{\boldsymbol{\varphi}}_1 = \begin{bmatrix} \nabla \hat{\varphi}_1 & 0 & \nabla \hat{\varphi}_2 & 0 & \nabla \hat{\varphi}_3 & 0 & \nabla \hat{\varphi}_4 & 0 \end{bmatrix}$$

$$\nabla \hat{\boldsymbol{\varphi}}_2 = \begin{bmatrix} 0 & \nabla \hat{\varphi}_1 & 0 & \nabla \hat{\varphi}_2 & 0 & \nabla \hat{\varphi}_3 & 0 & \nabla \hat{\varphi}_4 \end{bmatrix}$$

$$\nabla \hat{\varphi}_i = \begin{Bmatrix} \frac{\partial \hat{\varphi}_i}{\partial \hat{x}_1} \\ \frac{\partial \hat{\varphi}_i}{\partial \hat{x}_2} \end{Bmatrix}$$

$$\hat{\boldsymbol{\psi}} = \begin{bmatrix} \frac{\partial \hat{\varphi}_1}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_1}{\partial \hat{x}_2} & \frac{\partial \hat{\varphi}_2}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_2}{\partial \hat{x}_2} & \frac{\partial \hat{\varphi}_3}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_3}{\partial \hat{x}_2} & \frac{\partial \hat{\varphi}_4}{\partial \hat{x}_1} & \frac{\partial \hat{\varphi}_4}{\partial \hat{x}_2} \end{bmatrix}$$

Elemento isoparamétrico Q1P0

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\hat{x}_1,\hat{x}_2)\, d\hat{x}_1 d\hat{x}_2 \approx \sum_{k=1}^{n} f(\hat{x}_1^k,\hat{x}_2^k) w_k$$

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial F_1}{\partial \hat{x}_1} & \dfrac{\partial F_1}{\partial \hat{x}_2} \\ \dfrac{\partial F_2}{\partial \hat{x}_1} & \dfrac{\partial F_2}{\partial \hat{x}_2} \end{bmatrix}$$

$$\boldsymbol{A}^{\mathrm{K\,VV}} = \mu \int_{\hat{\mathrm{K}}} \left(\left(\boldsymbol{J}^{-T}\nabla\hat{\boldsymbol{\varphi}}_1\right)^T \boldsymbol{J}^{-T}\nabla\hat{\boldsymbol{\varphi}}_1 + \left(\boldsymbol{J}^{-T}\nabla\hat{\boldsymbol{\varphi}}_2\right)^T \boldsymbol{J}^{-T}\nabla\hat{\boldsymbol{\varphi}}_2 \right) \left|\det \boldsymbol{J}\right| d\hat{\boldsymbol{x}} =$$

$$\approx \mu \sum_{k=1}^{n} \left[\left(\boldsymbol{J}^{-T}(\hat{\boldsymbol{x}}^k)\nabla\hat{\boldsymbol{\varphi}}_1(\hat{\boldsymbol{x}}^k)\right)^T \boldsymbol{J}^{-T}(\hat{\boldsymbol{x}}^k)\nabla\hat{\boldsymbol{\varphi}}_1(\hat{\boldsymbol{x}}^k) + \left(\boldsymbol{J}^{-T}(\hat{\boldsymbol{x}}^k)\nabla\hat{\boldsymbol{\varphi}}_2(\hat{\boldsymbol{x}}^k)\right)^T \boldsymbol{J}^{-T}(\hat{\boldsymbol{x}}^k)\nabla\hat{\boldsymbol{\varphi}}_2(\hat{\boldsymbol{x}}^k) \right] w_k \left|\det \boldsymbol{J}(\hat{\boldsymbol{x}}^k)\right|$$

$$\boldsymbol{A}^{\mathrm{K\,VP}} = -\int_{\hat{\mathrm{K}}} \underbrace{\begin{bmatrix} \boldsymbol{J}^{-T} & & & \\ & \boldsymbol{J}^{-T} & & \\ & & \boldsymbol{J}^{-T} & \\ & & & \boldsymbol{J}^{-T} \end{bmatrix}}_{\boldsymbol{M}} \hat{\boldsymbol{\psi}}^T \left|\det \boldsymbol{J}\right| d\hat{\boldsymbol{x}} \approx -\sum_{k=1}^{n} \boldsymbol{M}(\hat{\boldsymbol{x}}^k)\, \hat{\boldsymbol{\psi}}^T(\hat{\boldsymbol{x}}^k)\, w_k \left|\det \boldsymbol{J}(\hat{\boldsymbol{x}}^k)\right|$$

$$\boldsymbol{A}^{K} = \begin{bmatrix} \boldsymbol{A}^{K\,VV} & \boldsymbol{A}^{K\,VP} \\ \boldsymbol{A}^{K\,PV} & \boldsymbol{0} \end{bmatrix}$$

Observaciones

1) Notar que en la formulación no se introdujo aún la restricción:

$$\int_{\Omega} q\, d\mathbf{x} = 0$$

en el campo H. Esta restricción es global, y abarca todo el dominio. Se puede introducir a través de un multiplicador de Lagrange adicional una vez ensambladas las matrices de todos los elementos.

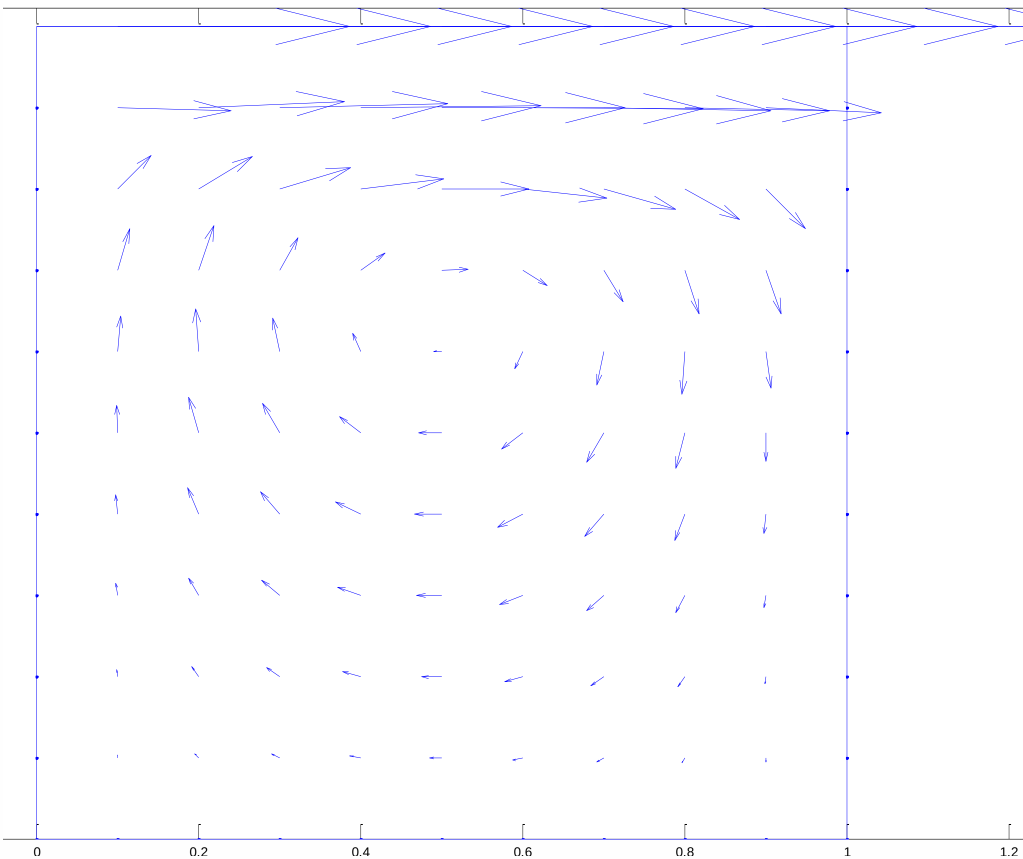
2) Notar además que es necesario fijar en algún punto el valor de la presión. Se recomienda fijar la presión en cero, por ejemplo, en la esquina inferior izquierda.

3) Se dan los resultados obtenidos para el problema de la cavidad cuadrada en las próximas transparencias. Se comparan con resultados de velocidad para elemento P5. Notar que la predicción de la posición vértical del vórtice difiere levemente entre ambas (??).

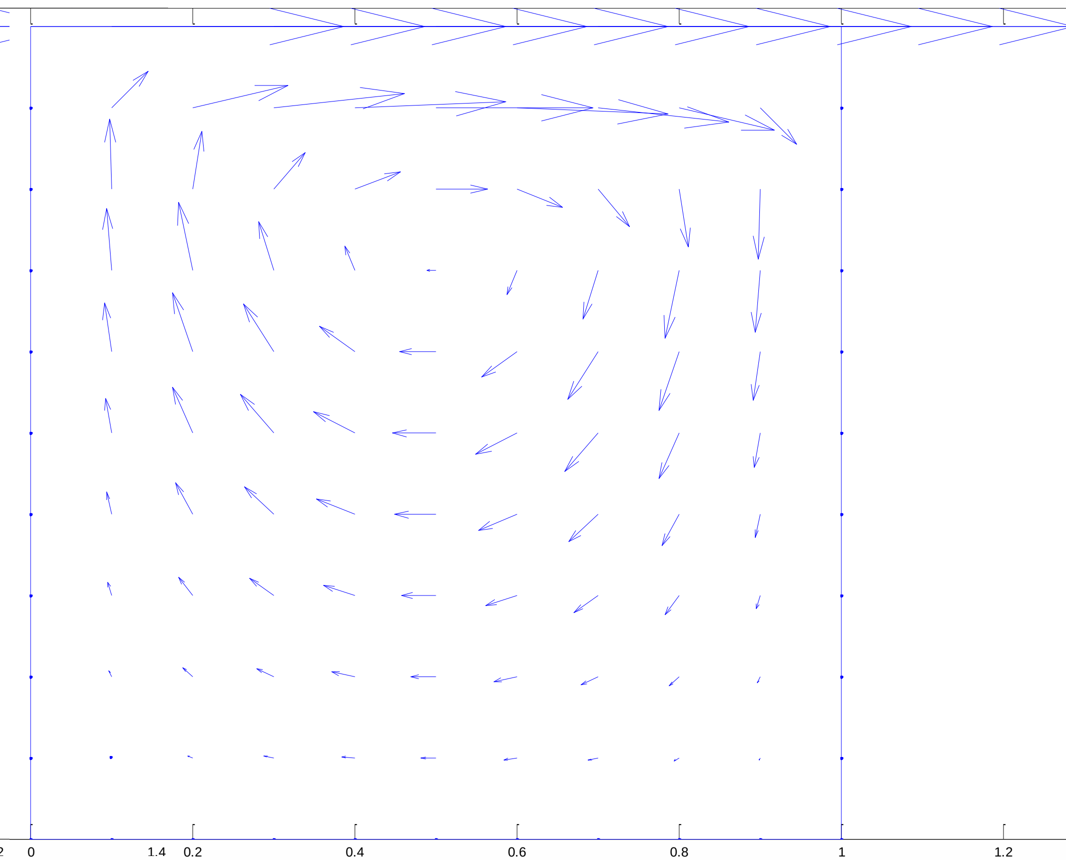
4) Notar las oscilaciones de presión propias de la formulación. De no aplicarse la restricción (1) los resultados de presión son completamente oscilatorios.

Ejemplo: Cavidad Cuadrada

Malla 10x10x2 triángulos P5

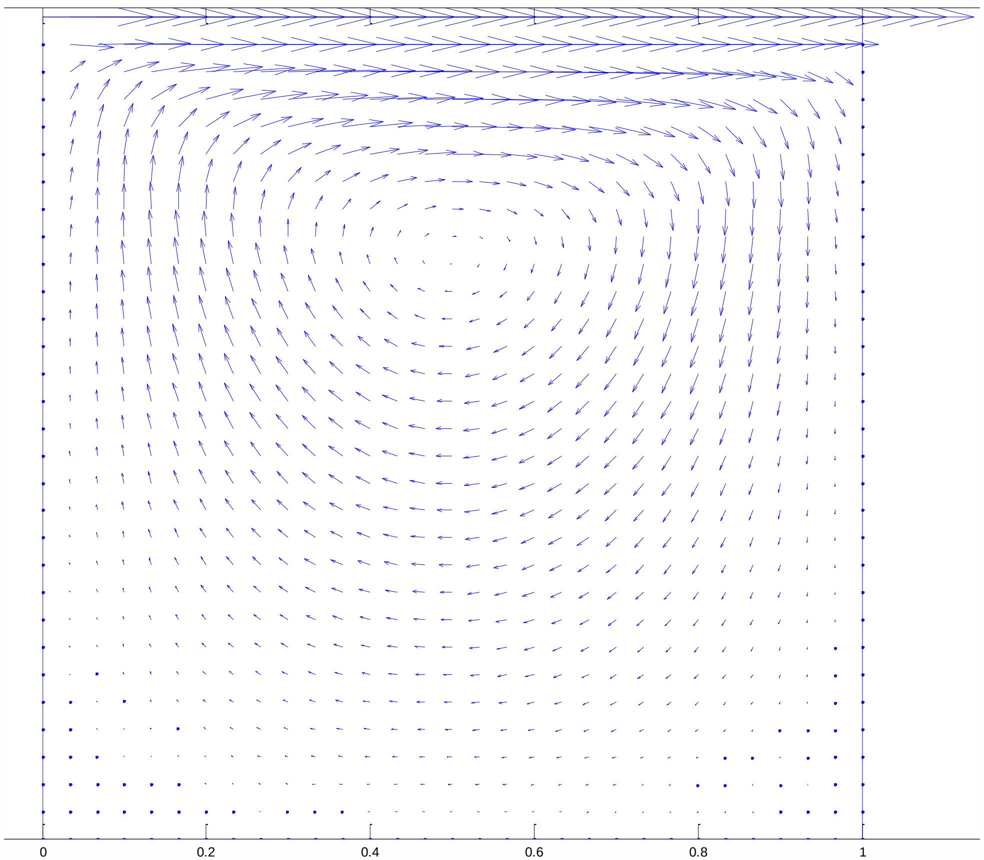


Malla 10x10 rectángulos Q1P0

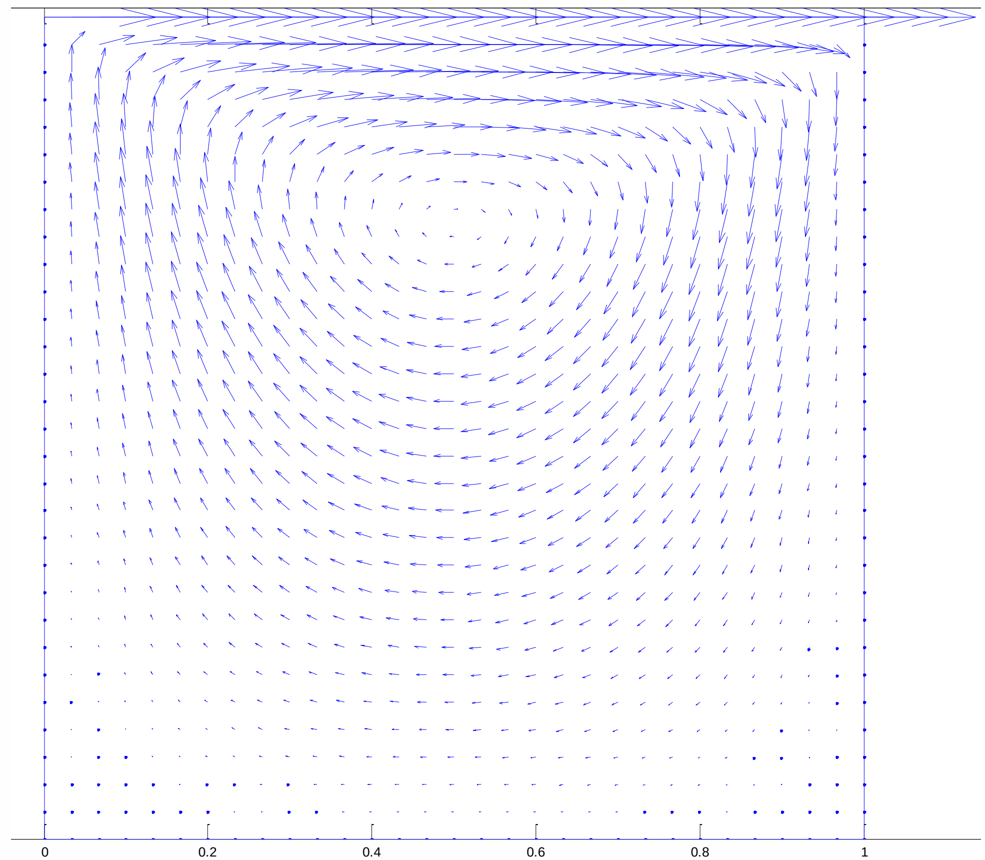


Ejemplo: Cavidad Cuadrada

Malla 30x30x2 triángulos P5

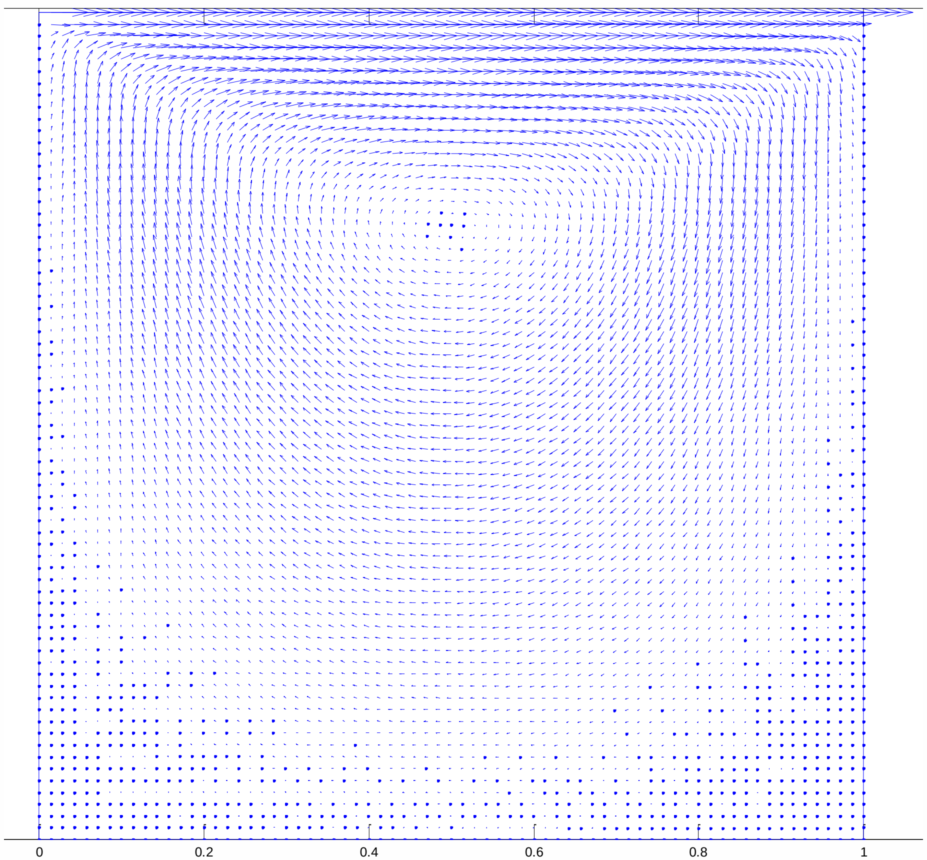


Malla 30x30 rectángulos Q1P0

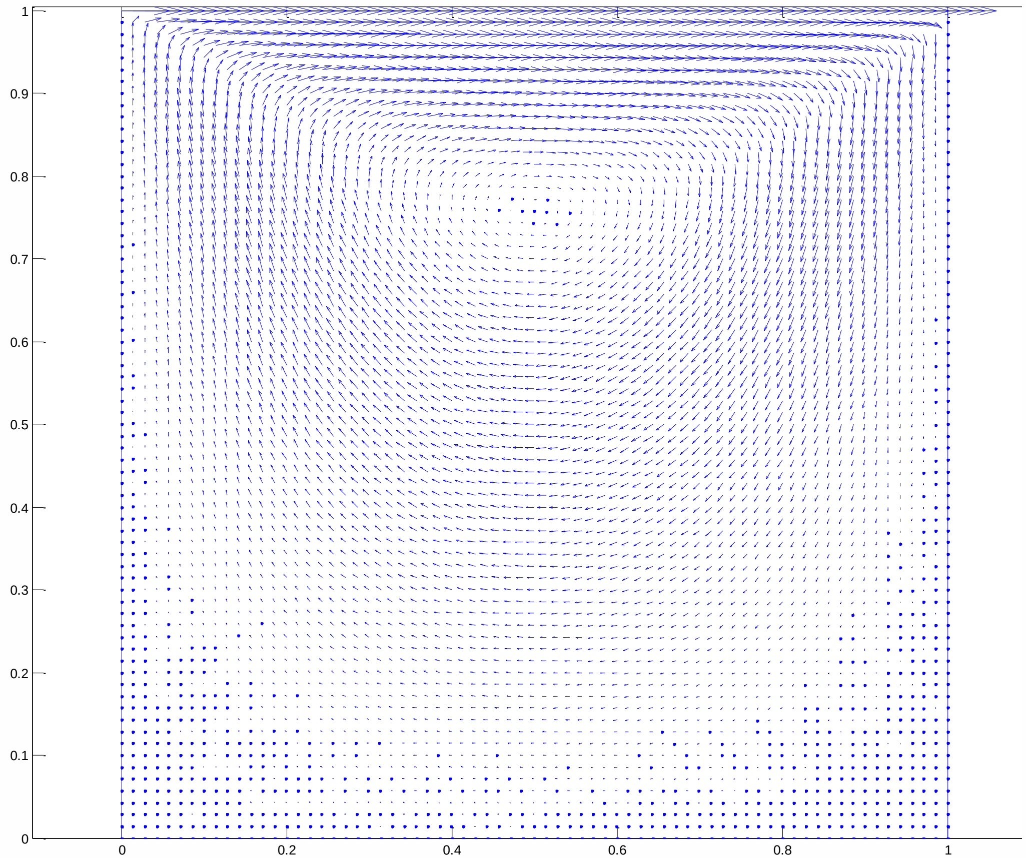


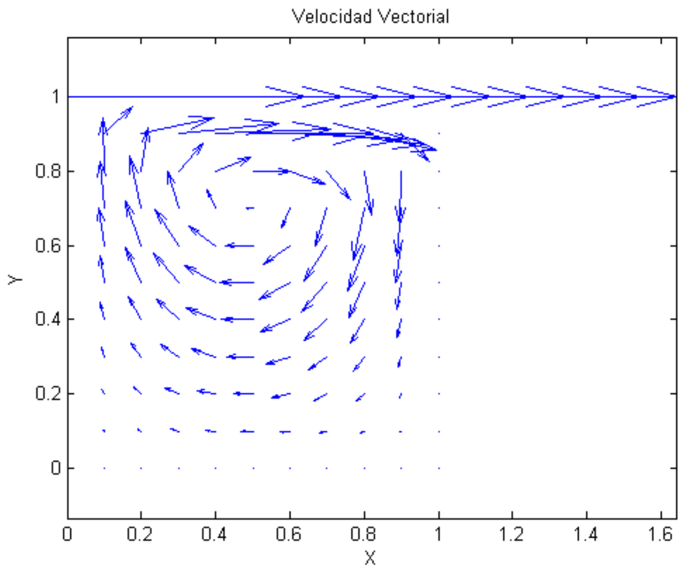
Ejemplo: Cavidad Cuadrada

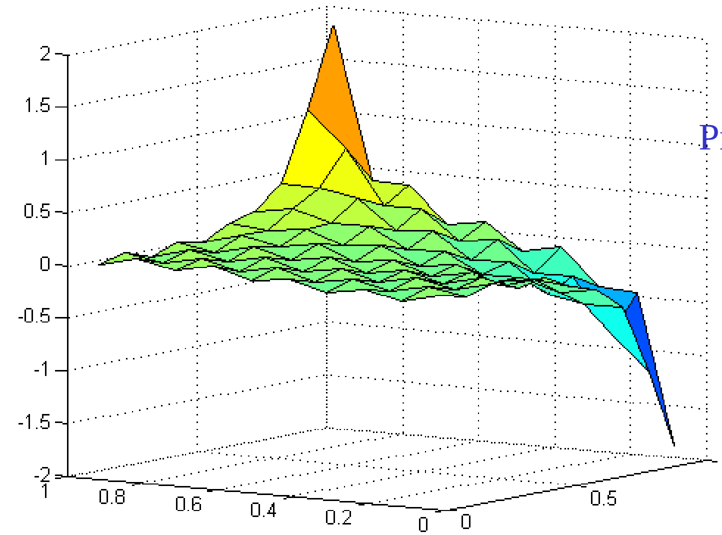
Malla 70x70x2 triángulos P5



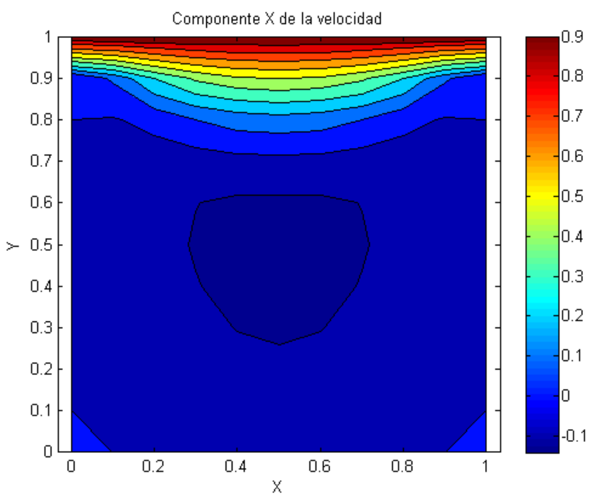
Malla 70x70 rectángulos Q1P0

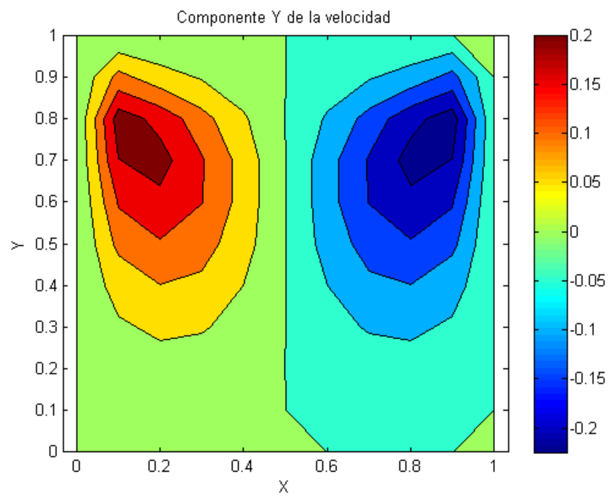


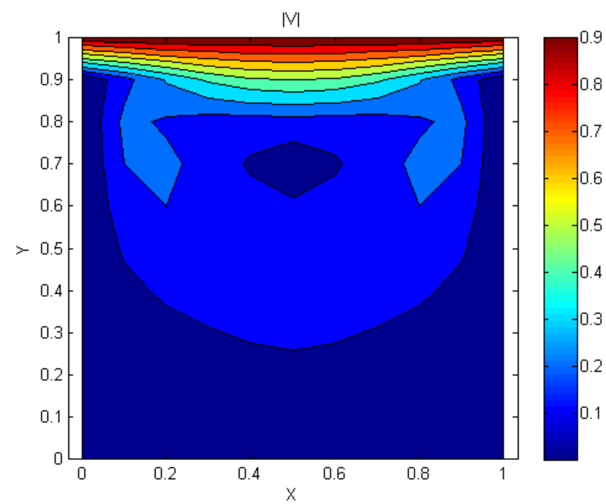

Velocidad Vectorial
Y
X
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2

Presión


Componente X de la velocidad
Y
X


Componente Y de la velocidad
Y
X


|V|
Y
X