

# Introducción al Método de los Elementos Finitos

## Parte 5

### Algunas aplicaciones del MEF a problemas elípticos

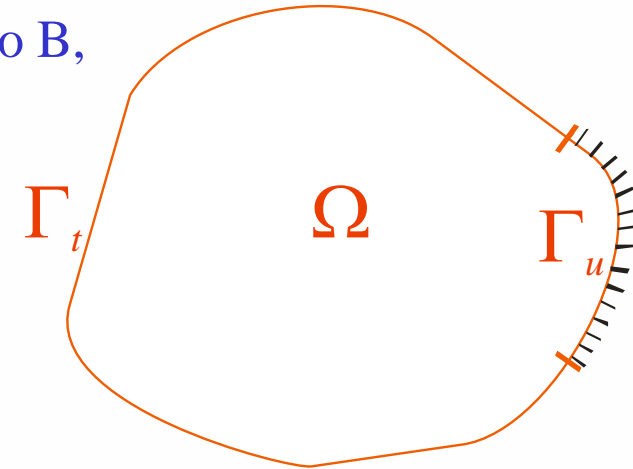
Alberto Cardona, Víctor Fachinotti  
Cimec (UNL/Conicet), Santa Fe, Argentina

25/Oct/16

# Algunas aplicaciones del MEF a problemas elípticos

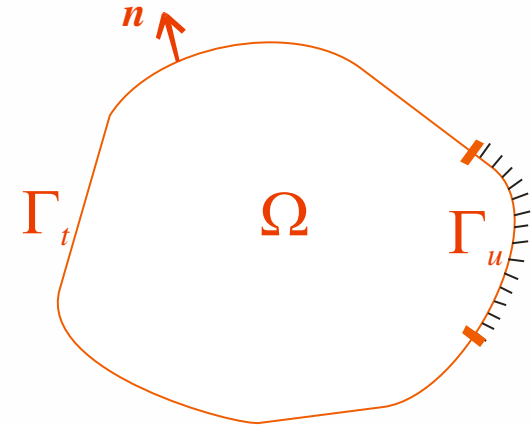
## Problema de Elasticidad

- Consideremos un cuerpo elástico, isótropo y homogéneo  $B$ , que ocupa el dominio acotado  $\Omega \in \mathbb{R}^3$ , con frontera  $\Gamma = \Gamma_t \cup \Gamma_u$  tal que  $\Gamma_t \cap \Gamma_u = \emptyset$  y área  $\Gamma_u > 0$ .
- El cuerpo  $B$  está sometido a una fuerza volumétrica  $f$ , y a una fuerza superficial  $t$  aplicada sobre  $\Gamma_t$ .
- Se supone  $B$  fijo a lo largo de  $\Gamma_u$ .
- Se busca determinar
  - desplazamientos  $u$ .
  - deformaciones  $\varepsilon$ , dependientes de los desplazamientos de acuerdo a la cinemática de la deformación.
  - tensiones  $\sigma$ , dependientes de las deformaciones de acuerdo a la ley constitutiva del material.



## Problema de Elasticidad

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} & \text{en } \Omega & \text{Ecuación de equilibrio} \\ \mathbf{u} = \mathbf{0} & \text{sobre } \Gamma_u & \text{CB Dirichlet (despl. impuesto)} \\ \boldsymbol{\sigma} \mathbf{n} = \mathbf{t} & \text{sobre } \Gamma_t & \text{CB Neumann (tracción impuesta)} \end{cases}$$



- Ecuaciones de clausura

- Cinemáticas: asumiendo pequeñas deformaciones:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

- Constitutivas: asumiendo comportamiento elástico lineal (ley de Hooke):

$$\sigma_{ij} = \lambda \sum_{k=1}^3 \varepsilon_{kk} \delta_{ij} + \mu \varepsilon_{ij}, \quad \lambda, \mu \in \mathbb{R}^+, \text{ ctes. de Lamé.}$$

$$\text{con } \mu = \frac{E}{1+\nu}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \left\{ \begin{array}{l} E : \text{módulo de Elasticidad} \\ \nu : \text{coeficiente de Poisson} \end{array} \right.$$

- Nota:** en adelante, usaremos las siguientes convenciones de notación:

- derivada parcial:  $u_{i,j} = \frac{\partial u_i}{\partial x_j}$

- sumatoria:  $\sigma_{ij} n_j = \sum_{j=1}^3 \sigma_{ij} n_j$

# Forma variacional del problema de Elasticidad

- Dado  $(D) \quad \sigma_{ij,j} + f_i = 0 \quad \text{en } \Omega, \quad i, j = 1, 2, 3$   
se llega a  $(V) \quad \text{Hallar } \mathbf{u} \in V / a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}), \quad \forall \mathbf{v} \in V$   
con  $V = \left\{ \mathbf{v} : \mathbf{v} \in [H^1(\Omega)]^3 \text{ y } \mathbf{v} = \mathbf{0} \text{ en } \Gamma_u \right\}$

haciendo

$$\begin{aligned}
 & \underbrace{\int_{\Omega} \sigma_{ij,j} v_i dx}_{\text{T. de Green}} + \int_{\Omega} f_i v_i dx = 0 \\
 & - \int_{\Omega} \sigma_{ij} v_{i,j} dx + \int_{\Gamma} \sigma_{ij} n_j v_i ds + \int_{\Omega} f_i v_i dx = 0 \\
 & - \int_{\Omega} \frac{\sigma_{ij} v_{i,j} + \sigma_{ji} v_{j,i}}{2} dx + \int_{\Gamma_t} t_i v_i ds + \int_{\Omega} f_i v_i dx = 0 \quad \text{CB} \\
 & \underbrace{- \int_{\Omega} \sigma_{ij} \frac{v_{i,j} + v_{j,i}}{2} dx}_{\text{Cinemática de pequeñas deformaciones}} + \int_{\Gamma_t} t_i v_i ds + \int_{\Omega} f_i v_i dx = 0 \quad \text{Simetría de } \sigma \\
 & - \int_{\Omega} \sigma_{ij} \varepsilon_{ij}(\mathbf{v}) dx + \int_{\Gamma_t} t_i v_i ds + \int_{\Omega} f_i v_i dx = 0 \\
 & - \int_{\Omega} \left[ \lambda u_{i,i} \delta_{ij} + \mu \varepsilon_{ij}(\mathbf{u}) \right] \varepsilon_{ij}(\mathbf{v}) dx + \int_{\Gamma_t} t_i v_i ds + \int_{\Omega} f_i v_i dx = 0 \\
 & \underbrace{\hspace{10em}}_{a(\mathbf{u}, \mathbf{v})} \quad \underbrace{\hspace{10em}}_{L(\mathbf{v})}
 \end{aligned}$$

## Forma variacional del problema de Elasticidad (cont.)

- Se puede demostrar que la forma lineal  $L(\mathbf{v}) = \int_{\Omega} f_i v_i dx + \int_{\Gamma_t} t_i v_i ds$  es continua, i.e.,  $L(\mathbf{v}) \leq \lambda \|\mathbf{v}\|_V$ ,  $\forall \mathbf{v} \in V$ ,  $\lambda \in \mathbb{R}^+$ .
- Se puede demostrar que la forma bilineal
 
$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left[ \lambda u_{k,k} \delta_{ij} + \mu \varepsilon_{ij}(\mathbf{u}) \right] \varepsilon_{ij}(\mathbf{v}) dx = \int_{\Omega} \left[ \lambda \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} + \mu \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \right] dx$$
  - es simétrica, i.e.,  $a(\mathbf{u}, \mathbf{v}) = a(\mathbf{v}, \mathbf{u})$ ,  $\forall \mathbf{u}, \mathbf{v} \in V$ .
  - continua, i.e.,  $|a(\mathbf{u}, \mathbf{v})| \leq \gamma \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}$ ,  $\forall \mathbf{u}, \mathbf{v} \in V$ ,  $\gamma \in \mathbb{R}^+$ .
  - V-elíptica, i.e.,  $a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|_{H^1(\Omega)}^2$ ,  $\forall \mathbf{v} \in V$ ,  $\alpha \in \mathbb{R}^+$ , con  $\|\mathbf{v}\|_{H^1(\Omega)}^2 = \sum_{i=1}^3 \|v_i\|_{H^1(\Omega)}^2$ .

Demo.:  $a(\mathbf{v}, \mathbf{v}) = \lambda \int_{\Omega} (\operatorname{div} \mathbf{v})^2 dx + \mu \int_{\Omega} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) dx$

$$\geq \mu \int_{\Omega} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) dx \geq \mu c \|\mathbf{v}\|_{H^1(\Omega)}^2, \quad c \in \mathbb{R}^+.$$

Desigualdad de Korn

## MEF aplicado al problema de Elasticidad

- Consideremos el problema de Elasticidad en  $\Omega \subset \mathbb{R}^2$ .
- Sea  $T_h = \{K\}$  una malla de triángulos de  $\Omega$ . Definimos el espacio de EF

$$V_h = \left\{ \mathbf{v} : \mathbf{v} \in V \text{ y } \mathbf{v}|_K \in [P_1(K)]^2, \forall K \in T_h \right\}$$

- El MEF aplicado al problema de Elasticidad consiste en

$$\text{Hallar } \mathbf{u}_h \in V_h / a(\mathbf{u}_h, \mathbf{v}) = L(\mathbf{v}), \quad \forall \mathbf{v} \in V_h$$

- La solución  $\mathbf{u}_h \in V_h$  satisface

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} \leq Ch \|\mathbf{u}\|_{H^2(\Omega)}$$

## Funciones de base para el triángulo lineal

Toda función  $\mathbf{v} \in [P_1(\mathbb{K})]^2$  puede representarse

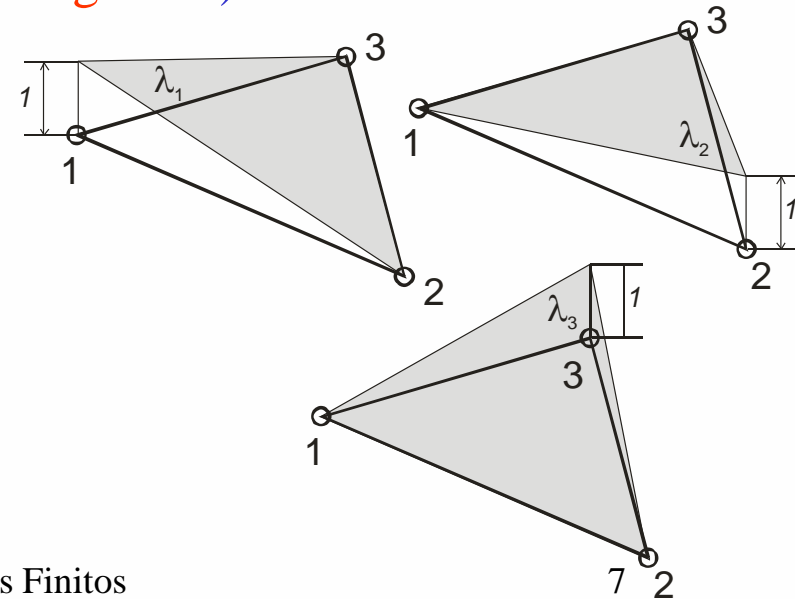
$$\mathbf{v}(x, y) = \begin{Bmatrix} v_x(x, y) \\ v_y(x, y) \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^3 N_i(x, y) V_i^x \\ \sum_{i=1}^3 N_i(x, y) V_i^y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} V_1^x \\ V_1^y \\ V_2^x \\ V_2^y \\ V_3^x \\ V_3^y \end{Bmatrix}, \quad \forall (x, y) \in \mathbb{K}.$$

Las funciones de base  $N_i = \lambda_i$  ( $\equiv$  coord de área del triángulo  $\mathbb{K}$ ) resultan:

$$N_i(x, y) = a_i + b_i x + c_i y$$

con:

$$\begin{aligned} a_1 &= \frac{x_2 y_3 - x_3 y_2}{2 A_K} & b_1 &= \frac{y_2 - y_3}{2 A_K} & c_1 &= \frac{x_3 - x_2}{2 A_K} \\ a_2 &= \frac{x_3 y_1 - x_1 y_3}{2 A_K} & b_2 &= \frac{y_3 - y_1}{2 A_K} & c_2 &= \frac{x_1 - x_3}{2 A_K} \\ a_3 &= \frac{x_1 y_2 - x_2 y_1}{2 A_K} & b_3 &= \frac{y_1 - y_2}{2 A_K} & c_3 &= \frac{x_2 - x_1}{2 A_K} \end{aligned}$$



## MEF aplicado al problema de Elasticidad

Notar:

$$a(\mathbf{v}, \mathbf{u}_h) = \int_{\Omega} \left[ \lambda \varepsilon_{jj}(\mathbf{v}) \varepsilon_{ii}(\mathbf{u}_h) + \mu \varepsilon_{ij}(\mathbf{v}) \varepsilon_{ij}(\mathbf{u}_h) \right] dx = \int_{\Omega} \begin{Bmatrix} \varepsilon_{xx}(\mathbf{v}) \\ \varepsilon_{yy}(\mathbf{v}) \\ 2\varepsilon_{xy}(\mathbf{v}) \end{Bmatrix}^T \begin{bmatrix} \lambda + \mu & \lambda & 0 \\ \lambda & \lambda + \mu & 0 \\ 0 & 0 & \frac{\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}(\mathbf{u}_h) \\ \varepsilon_{yy}(\mathbf{u}_h) \\ 2\varepsilon_{xy}(\mathbf{u}_h) \end{Bmatrix} dx$$

El vector de deformación :

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \begin{Bmatrix} \varepsilon_{xx}(\mathbf{v}) \\ \varepsilon_{yy}(\mathbf{v}) \\ 2\varepsilon_{xy}(\mathbf{v}) \end{Bmatrix} = \begin{Bmatrix} \frac{\partial v_x}{\partial x} \\ \frac{\partial v_y}{\partial y} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} V_1^x \\ V_1^y \\ V_2^x \\ V_2^y \\ V_3^x \\ V_3^y \end{Bmatrix} = \mathbf{B} \mathbf{V}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}_h) = \mathbf{B} \mathbf{U}_h$$

Para el triángulo lineal, la matriz  $\mathbf{B}$  es constante :

$$\mathbf{B} = \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$



# MEF aplicado al problema de Elasticidad

Luego:

$$a(\mathbf{v}, \mathbf{u}_h) = \sum_K a_K(\mathbf{v}, \mathbf{u}_h)$$

con

$$a_K(\mathbf{v}, \mathbf{u}_h) = \int_K \mathbf{V}^T \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{U}_h dx = \mathbf{V}^T \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{U}_h A_K$$

$$\mathbf{D} = \begin{bmatrix} \lambda + \mu & \lambda & 0 \\ \lambda & \lambda + \mu & 0 \\ 0 & 0 & \mu/2 \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}$$

(estado *plano de deformación*)

En consecuencia, la matriz de rigidez elemental :

$$\mathbf{A}_K = \mathbf{B}^T \mathbf{D} \mathbf{B} A_K$$

Notar:

$$\sigma_{ij} = \lambda \sum_{k=1}^3 \varepsilon_{kk} \delta_{ij} + \mu \varepsilon_{ij} \quad \rightarrow \quad \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \lambda + \mu & \lambda & 0 \\ \lambda & \lambda + \mu & 0 \\ 0 & 0 & \mu/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \mathbf{D} \boldsymbol{\varepsilon}$$

$$a(u_h, v) = L(v) \quad \forall v \in V_h$$

$$\int \mathcal{N} f dx$$

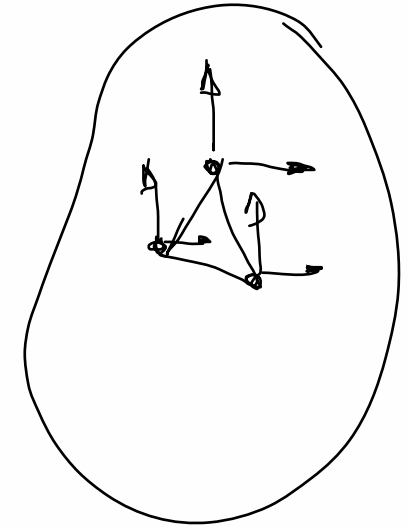
$$\sum_k V^T B^T D B A_k U = V^T F$$

' Variables

$$\left( \sum_k B^T D B A_k \right) U = F$$

$v = \sum \mathcal{N} V$

K matriz de rigidez



## Problema de Stokes

- Consideremos las ecuaciones de Stokes para el flujo estacionario de un fluido Newtoniano incompresible encerrado en un dominio  $\Omega \subset \mathbb{R}^3$ , sometido a una fuerza volumétrica  $f$ :

$$\sigma_{ij,j} + f_i = 0 \quad \text{en } \Omega, \quad \text{Balance de cant. de movto.}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij}(\mathbf{u}) - p\delta_{ij} \quad \text{en } \Omega, \quad \text{Ley const. de fluido Newtoniano}$$

$$u_{i,i} = 0 \quad \text{en } \Omega, \quad \text{Condición de incompresibilidad}$$

$$u_i = 0 \quad \text{sobre } \Gamma, \quad \text{CB Dirichlet}$$

$\mathbf{u}$ : velocidad

$\boldsymbol{\sigma}$ : tensión

$p$ : presión

$\mu$ : viscosidad

$$-\mu\Delta u_i + p_{,i} = f_i \quad \text{en } \Omega, \quad \text{Balance de cant. de movto. p/fluido Newtoniano}$$

- Definimos el espacio de funciones de prueba

$$V = \left\{ \mathbf{v} : \mathbf{v} \in [H_0^1(\Omega)]^3 \text{ y } \operatorname{div} \mathbf{v} = 0 \text{ en } \Omega \right\}$$

- Luego, podemos llevar el problema de Stokes a la forma variacional

$$(V) \quad \text{Hallar } \mathbf{u} \in V / a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}), \quad \forall \mathbf{v} \in V$$

## Forma variacional del problema de Stokes

- Para llevar el problema de Stokes a la forma variacional hacemos

$$f_i = -\Delta u_i + p_{,i}$$

$$\int_{\Omega} f_i v_i dx = -\mu \int_{\Omega} \Delta u_i v_i dx + \int_{\Omega} p_{,i} v_i dx$$

$$\int_{\Omega} f_i v_i dx = \mu \int_{\Omega} \nabla u_i \cdot \nabla v_i dx - \underbrace{\mu \int_{\Gamma} \frac{\partial u_i}{\partial n} v_i ds}_{=0} - \underbrace{\int_{\Omega} p v_{i,i} dx}_{=0} + \underbrace{\int_{\Gamma} p n_i v_i ds}_{=0}$$

$$\underbrace{\int_{\Omega} f_i v_i dx}_{L(\mathbf{v})} = \underbrace{\mu \int_{\Omega} \nabla u_i \cdot \nabla v_i dx}_{a(\mathbf{u}, \mathbf{v})}$$

- Dado que  $\mu > 0$ , se demuestra (ídem problema de Poisson) que  $a(\cdot, \cdot)$  es simétrica, continua y V-elíptica.
- Se demuestra también (ídem problema de Poisson) que  $L(\cdot)$  es continua.
- Nota:** al adoptar un espacio de velocidades de divergencia nula, la formulación variacional no involucra la presión.

## Elemento P5 - MEF aplicado al problema de Stokes

Consideremos el problema de Stokes en  $\Omega \subset \mathbb{R}^2$ . Luego:

$$\mathbf{V} = \left\{ \mathbf{v} : \mathbf{v} = (v_1, v_2) \in [\mathbf{H}_0^1(\Omega)]^2 \text{ y } \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \text{ en } \Omega \right\}$$

- Si  $\Omega$  es simplemente conexo (i.e., no contiene agujeros),  $\text{div } \mathbf{v} = 0$  en  $\Omega$  si y solo si

$$\mathbf{v} = \left( \frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right) = \text{rot } \varphi \text{ para alguna función } \varphi.$$

$\varphi$ : función de corriente del campo de velocidades  $\mathbf{v}$ .

o sea:  $\mathbf{v} \in \mathbf{V} \Leftrightarrow \mathbf{v} = \text{rot } \varphi, \varphi \in \mathbf{H}_0^2(\Omega)$ .

- Adoptamos luego un subespacio  $W_h$  de dimensión finita de  $\mathbf{H}_0^2(\Omega)$  (usamos por ej. el elemento finito  $C^1$ -continuo ya visto) y definimos  $\mathbf{V}_h = \{ \mathbf{v} : \mathbf{v} = \text{rot } \varphi, \varphi \in W_h \}$ .
- Se formula el MEF reemplazando  $\mathbf{V}$  por  $\mathbf{V}_h \subset \mathbf{V}$  en la formulación variacional. La solución  $\mathbf{u}_h \in \mathbf{V}_h$  satisface

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} \leq Ch^4 |\mathbf{u}|_{\mathbf{H}^5(\Omega)}$$

$$\int \nabla u_i \cdot \nabla v_i \, d\mathbf{x} = \int u_{i,j} \cdot v_{i,j} \, d\mathbf{x}$$

$$\mathbf{u} = \text{rot}(\varphi \mathbf{e}_3) = \text{rot} \begin{Bmatrix} 0 \\ 0 \\ \varphi \end{Bmatrix} \quad \text{rot } \mathbf{t} \Rightarrow \varepsilon_{ijk} t_{k,j} \quad \longrightarrow \quad u_i = \varepsilon_{ij3} \varphi_{,j} \quad \longrightarrow \quad \mathbf{u} = \begin{Bmatrix} \varphi_{,2} \\ -\varphi_{,1} \\ 0 \end{Bmatrix}$$

$$u_{i,j} = \varepsilon_{ik3} \varphi_{,kj}$$

$$u_{i,j} v_{i,j} = \varepsilon_{ik3} \varphi_{,kj} \varepsilon_{il3} \psi_{,lj} = (\delta_{kl} \delta_{33} - \delta_{k3} \delta_{l3}) \varphi_{,kj} \psi_{,lj} = \varphi_{,lj} \psi_{,lj} - \varphi_{,3j} \psi_{,3j}$$

$$\int u_{i,j} v_{i,j} \, d\mathbf{x} = \int \varphi_{,1j} \psi_{,1j} + \varphi_{,2j} \psi_{,2j} \, d\mathbf{x}$$

## Elemento P5 - Desarrollo del MEF para Stokes

Desarrollaremos un EF triangular C1-continuo. El campo  $\varphi(x, y)$  será aproximado por polinomios de grado 5:

$$\varphi(x, y) = \sum_{0 \leq i+j \leq 5} c_{ij} x'^i y'^j = c_{00} + c_{10}x' + c_{01}y' + c_{20}x'^2 + c_{11}x'y' + c_{02}y'^2 + \dots = \mathbf{p}^T(x, y)\mathbf{c}, \quad \forall (x, y) \in K.$$

con:

$$x' = \frac{x - x_0}{L}; \quad y' = \frac{y - y_0}{L}; \quad L \quad : \text{longitud característica de elemento (ver adelante)}$$

$(x_0, y_0)$  : centroide del elemento

$$\mathbf{p}^T(x, y) = [1 \quad x' \quad y' \quad x'^2 \quad x'y' \quad y'^2 \quad x'^3 \quad \dots]$$

$$\mathbf{c} = [c_{00} \quad c_{10} \quad c_{01} \quad c_{20} \quad c_{11} \quad c_{02} \quad c_{30} \quad c_{21} \quad c_{12} \quad c_{03} \quad c_{40} \quad c_{31} \quad c_{22} \quad c_{13} \quad \dots \quad c_{23} \quad c_{14} \quad c_{05}]^T$$

Para el cálculo de los coeficientes  $\mathbf{c}$  expresamos el valor del campo en los nodos

$$\varphi(x_k, y_k) = \sum_{0 \leq i+j \leq 5} c_{ij} x_k'^i y_k'^j = c_{00} + c_{10}x_k' + c_{01}y_k' + c_{20}x_k'^2 + c_{11}x_k'y_k' + c_{02}y_k'^2 + \dots = \varphi^k, \quad k = 1, 2, 3.$$

Obtenemos así tres ecuaciones con 21 incógnitas:

$$\begin{bmatrix} 1 & x_k' & y_k' & x_k'^2 & x_k'y_k' & y_k'^2 & x_k'^3 & x_k'^2 y_k' & x_k'y_k'^2 & y_k'^3 & x_k'^4 & x_k'^3 y_k' & x_k'^2 y_k'^2 & x_k'y_k'^3 & y_k'^4 & x_k'^5 & x_k'^4 y_k' & x_k'^3 y_k'^2 & x_k'^2 y_k'^3 & x_k'y_k'^4 & y_k'^5 \end{bmatrix} \mathbf{c} = \varphi^k$$

$k = 1, 2, 3.$

## Elemento P5 - Desarrollo del MEF para Stokes

Calculando luego el valor de la derivada respecto de  $x$  en los nodos, logramos tres ecuaciones más:

$$\varphi_{,x}(x_k, y_k) = \sum_{0 \leq i+j \leq 5} c_{ij} \frac{i}{L} x_k^{i-1} y_k^j = \frac{1}{L} c_{10} + \frac{2}{L} c_{20} x'_k + \frac{1}{L} c_{11} y'_k + \dots = \varphi_{,x}^k, \quad k = 1, 2, 3.$$

$$\begin{bmatrix} 0 & 1 & 0 & 2x'_k & y'_k & 0 & 3x_k'^2 & 2x'_k y'_k & y_k'^2 & 0 & 4x_k'^3 & 3x_k'^2 y'_k & 2x'_k y_k'^2 & y_k'^3 & 0 & 5x_k'^4 & 4x_k'^3 y'_k & 3x_k'^2 y_k'^2 & 2x'_k y_k'^3 & y_k'^4 & 0 \end{bmatrix} \mathbf{c} = L \varphi_{,x}^k$$

$k = 1, 2, 3.$

Repitiendo el proceso para la evaluación de todos los grados de libertad nodales, obtenemos un conjunto de 18 ecuaciones con 21 incógnitas.

Este proceso puede hacerse usando un programa de manipulación simbólica, como se indica en el siguiente slide.



# Elemento P5 - Desarrollo del MEF para Stokes

## Programa para calculo simbólico coeficientes

```
clear
syms C x y x1 y1 x2 y2 x3 y3 L
C = 1;
for i=1:5
    for j=0:i
        C = [C (x/L)^(i-j)*(y/L)^(j)];
    end
end

Cx = diff(C,x);
Cy = diff(C,y);
Cxx = diff(C,x,2);
Cxy = diff(Cx,y);
Cyy = diff(C,y,2);

% NODO 1
% phi(x1,y1) = phi_1
A(1,:) = subs(C, {x,y}, {x1,y1});
% dphi/dx (x1,y1) = dphix_1
A(2,:) = subs(Cx, {x,y}, {x1,y1});
% dphi/dy (x1,y1) = dphiy_1
A(3,:) = subs(Cy, {x,y}, {x1,y1});

% d2phi/dx2 (x1,y1) = dphixx_1
A(4,:) = subs(Cxx, {x,y}, {x1,y1});
% d2phi/dxdy (x1,y1) = dphixy_1
A(5,:) = subs(Cxy, {x,y}, {x1,y1});
% d2phi/dy2 (x1,y1) = dphiyy_1
A(6,:) = subs(Cyy, {x,y}, {x1,y1});

% NODO 2
A(7,:) = subs(C, {x,y}, {x2,y2});
A(8,:) = subs(Cx, {x,y}, {x2,y2});
A(9,:) = subs(Cy, {x,y}, {x2,y2});
A(10,:) = subs(Cxx, {x,y}, {x2,y2});
A(11,:) = subs(Cxy, {x,y}, {x2,y2});
A(12,:) = subs(Cyy, {x,y}, {x2,y2});

% NODO 3
A(13,:) = subs(C, {x,y}, {x3,y3});
A(14,:) = subs(Cx, {x,y}, {x3,y3});
A(15,:) = subs(Cy, {x,y}, {x3,y3});
A(16,:) = subs(Cxx, {x,y}, {x3,y3});
A(17,:) = subs(Cxy, {x,y}, {x3,y3});
A(18,:) = subs(Cyy, {x,y}, {x3,y3});
```

Tenemos así 18 ecuaciones y 21 incógnitas:

$$\begin{bmatrix}
 1 & x'_1 & y'_1 & x'^2_1 & x'_1 y'_1 & y'^2_1 & x'^3_1 & x'^2_1 y'_1 & x'_1 y'^2_1 & y'^3_1 & x'^4_1 & x'^3_1 y'_1 & x'^2_1 y'^2_1 & x'_1 y'^3_1 & y'^4_1 & x'^5_1 & x'^4_1 y'_1 & x'^3_1 y'^2_1 & x'^2_1 y'^3_1 & x'_1 y'^4_1 & y'^5_1 \\
 0 & 1 & 0 & 2x'_1 & y'_1 & 0 & 3x'^2_1 & 2x'_1 y'_1 & y'^2_1 & 0 & 4x'^3_1 & 3x'^2_1 y'_1 & 2x'_1 y'^2_1 & y'^3_1 & 0 & 5x'^4_1 & 4x'^3_1 y'_1 & 3x'^2_1 y'^2_1 & 2x'_1 y'^3_1 & y'^4_1 & 0 \\
 0 & 0 & 1 & 0 & x'_1 & 2y'_1 & 0 & x'^2_1 & 2x'_1 y'_1 & 3y'^2_1 & 0 & x'^3_1 & 2x'^2_1 y'_1 & 3x'_1 y'^2_1 & 4y'^3_1 & 0 & x'^4_1 & 2x'^3_1 y'_1 & 3x'^2_1 y'^2_1 & 4x'_1 y'^3_1 & 5y'^4_1 \\
 0 & 0 & 0 & 2 & 0 & 0 & 6x'_1 & 2y'_1 & 0 & 0 & 12x'^2_1 & 6x'_1 y'_1 & 2y'^2_1 & 0 & 0 & 20x'^3_1 & 12x'^2_1 y'_1 & 6x'_1 y'^2_1 & 2y'^3_1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x'_1 & 2y'_1 & 0 & 0 & 3x'^2_1 & 4x'_1 y'_1 & 3y'^2_1 & 0 & 0 & 4x'^3_1 & 6x'^2_1 y'_1 & 6x'_1 y'^2_1 & 4y'^3_1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x'_1 & 6y'_1 & 0 & 0 & 2x'^2_1 & 6x'_1 y'_1 & 12y'^2_1 & 0 & 0 & 2x'^3_1 & 6x'^2_1 y'_1 & 12x'_1 y'^2_1 & 20y'^3_1 \\
 1 & x'_2 & y'_2 & x'^2_2 & x'_2 y'_2 & y'^2_2 & x'^3_2 & x'^2_2 y'_2 & x'_2 y'^2_2 & y'^3_2 & x'^4_2 & x'^3_2 y'_2 & x'^2_2 y'^2_2 & x'_2 y'^3_2 & y'^4_2 & x'^5_2 & x'^4_2 y'_2 & x'^3_2 y'^2_2 & x'^2_2 y'^3_2 & x'_2 y'^4_2 & y'^5_2 \\
 0 & 1 & 0 & 2x'_2 & y'_2 & 0 & 3x'^2_2 & 2x'_2 y'_2 & y'^2_2 & 0 & 4x'^3_2 & 3x'^2_2 y'_2 & 2x'_2 y'^2_2 & y'^3_2 & 0 & 5x'^4_2 & 4x'^3_2 y'_2 & 3x'^2_2 y'^2_2 & 2x'_2 y'^3_2 & y'^4_2 & 0 \\
 0 & 0 & 1 & 0 & x'_2 & 2y'_2 & 0 & x'^2_2 & 2x'_2 y'_2 & 3y'^2_2 & 0 & x'^3_2 & 2x'^2_2 y'_2 & 3x'_2 y'^2_2 & 4y'^3_2 & 0 & x'^4_2 & 2x'^3_2 y'_2 & 3x'^2_2 y'^2_2 & 4x'_2 y'^3_2 & 5y'^4_2 \\
 0 & 0 & 0 & 2 & 0 & 0 & 6x'_2 & 2y'_2 & 0 & 0 & 12x'^2_2 & 6x'_2 y'_2 & 2y'^2_2 & 0 & 0 & 20x'^3_2 & 12x'^2_2 y'_2 & 6x'_2 y'^2_2 & 2y'^3_2 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x'_2 & 2y'_2 & 0 & 0 & 3x'^2_2 & 4x'_2 y'_2 & 3y'^2_2 & 0 & 0 & 4x'^3_2 & 6x'^2_2 y'_2 & 6x'_2 y'^2_2 & 4y'^3_2 & 0 \\
 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x'_2 & 6y'_2 & 0 & 0 & 2x'^2_2 & 6x'_2 y'_2 & 12y'^2_2 & 0 & 0 & 2x'^3_2 & 6x'^2_2 y'_2 & 12x'_2 y'^2_2 & 20y'^3_2 \\
 1 & x'_3 & y'_3 & x'^2_3 & x'_3 y'_3 & y'^2_3 & x'^3_3 & x'^2_3 y'_3 & x'_3 y'^2_3 & y'^3_3 & x'^4_3 & x'^3_3 y'_3 & x'^2_3 y'^2_3 & x'_3 y'^3_3 & y'^4_3 & x'^5_3 & x'^4_3 y'_3 & x'^3_3 y'^2_3 & x'^2_3 y'^3_3 & x'_3 y'^4_3 & y'^5_3 \\
 0 & 1 & 0 & 2x'_3 & y'_3 & 0 & 3x'^2_3 & 2x'_3 y'_3 & y'^2_3 & 0 & 4x'^3_3 & 3x'^2_3 y'_3 & 2x'_3 y'^2_3 & y'^3_3 & 0 & 5x'^4_3 & 4x'^3_3 y'_3 & 3x'^2_3 y'^2_3 & 2x'_3 y'^3_3 & y'^4_3 & 0 \\
 0 & 0 & 1 & 0 & x'_3 & 2y'_3 & 0 & x'^2_3 & 2x'_3 y'_3 & 3y'^2_3 & 0 & x'^3_3 & 2x'^2_3 y'_3 & 3x'_3 y'^2_3 & 4y'^3_3 & 0 & x'^4_3 & 2x'^3_3 y'_3 & 3x'^2_3 y'^2_3 & 4x'_3 y'^3_3 & 5y'^4_3 \\
 0 & 0 & 0 & 2 & 0 & 0 & 6x'_3 & 2y'_3 & 0 & 0 & 12x'^2_3 & 6x'_3 y'_3 & 2y'^2_3 & 0 & 0 & 20x'^3_3 & 12x'^2_3 y'_3 & 6x'_3 y'^2_3 & 2y'^3_3 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2x'_3 & 2y'_3 & 0 & 0 & 3x'^2_3 & 4x'_3 y'_3 & 3y'^2_3 & 0 & 0 & 4x'^3_3 & 6x'^2_3 y'_3 & 6x'_3 y'^2_3 & 4y'^3_3 & 0 \\
 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x'_3 & 6y'_3 & 0 & 0 & 2x'^2_3 & 6x'_3 y'_3 & 12y'^2_3 & 0 & 0 & 2x'^3_3 & 6x'^2_3 y'_3 & 12x'_3 y'^2_3 & 20y'^3_3
 \end{bmatrix}
 \mathbf{c} = \begin{cases}
 \varphi^1 \\
 L\varphi^1_{,x} \\
 L\varphi^1_{,y} \\
 L^2\varphi^1_{,xx} \\
 L^2\varphi^1_{,xy} \\
 L^2\varphi^1_{,yy} \\
 \varphi^2 \\
 L\varphi^2_{,x} \\
 L\varphi^2_{,y} \\
 L^2\varphi^2_{,xx} \\
 L^2\varphi^2_{,xy} \\
 L^2\varphi^2_{,yy} \\
 \varphi^3 \\
 L\varphi^3_{,x} \\
 L\varphi^3_{,y} \\
 L^2\varphi^3_{,xx} \\
 L^2\varphi^3_{,xy} \\
 L^2\varphi^3_{,yy}
 \end{cases}$$

## Elemento P5 - Desarrollo del MEF para Stokes

Las tres ecuaciones faltantes se obtienen calculando la derivada normal en los puntos medios de los lados:

$$\left[ \begin{array}{l}
 n_{12,x} \left( 0 \ 1 \ 0 \ 2x'_{12} \ y'_{12} \ 0 \ 3x'^2_{12} \ 2x'_{12}y'_{12} \ y'^2_{12} \ 0 \ 4x'^3_{12} \ 3x'^2_{12}y'_{12} \ 2x'_{12}y'^2_{12} \ y'^3_{12} \ 0 \ 5x'^4_{12} \ 4x'^3_{12}y'_{12} \ 3x'^2_{12}y'^2_{12} \ 2x'_{12}y'^3_{12} \ y'^4_{12} \ 0 \right) + \\
 n_{12,y} \left( 0 \ 0 \ 1 \ 0 \ x'_{12} \ 2y'_{12} \ 0 \ x'^2_{12} \ 2x'_{12}y'_{12} \ 3y'^2_{12} \ 0 \ x'^3_{12} \ 2x'^2_{12}y'_{12} \ 3x'_{12}y'^2_{12} \ 4y'^3_{12} \ 0 \ x'^4_{12} \ 2x'^3_{12}y'_{12} \ 3x'^2_{12}y'^2_{12} \ 4x'_{12}y'^3_{12} \ 5y'^4_{12} \right) \\
 n_{23,x} \left( 0 \ 1 \ 0 \ 2x'_{23} \ y'_{23} \ 0 \ 3x'^2_{23} \ 2x'_{23}y'_{23} \ y'^2_{23} \ 0 \ 4x'^3_{23} \ 3x'^2_{23}y'_{23} \ 2x'_{23}y'^2_{23} \ y'^3_{23} \ 0 \ 5x'^4_{23} \ 4x'^3_{23}y'_{23} \ 3x'^2_{23}y'^2_{23} \ 2x'_{23}y'^3_{23} \ y'^4_{23} \ 0 \right) + \\
 n_{23,y} \left( 0 \ 0 \ 1 \ 0 \ x'_{23} \ 2y'_{23} \ 0 \ x'^2_{23} \ 2x'_{23}y'_{23} \ 3y'^2_{23} \ 0 \ x'^3_{23} \ 2x'^2_{23}y'_{23} \ 3x'_{23}y'^2_{23} \ 4y'^3_{23} \ 0 \ x'^4_{23} \ 2x'^3_{23}y'_{23} \ 3x'^2_{23}y'^2_{23} \ 4x'_{23}y'^3_{23} \ 5y'^4_{23} \right) \\
 n_{31,x} \left( 0 \ 1 \ 0 \ 2x'_{31} \ y'_{31} \ 0 \ 3x'^2_{31} \ 2x'_{31}y'_{31} \ y'^2_{31} \ 0 \ 4x'^3_{31} \ 3x'^2_{31}y'_{31} \ 2x'_{31}y'^2_{31} \ y'^3_{31} \ 0 \ 5x'^4_{31} \ 4x'^3_{31}y'_{31} \ 3x'^2_{31}y'^2_{31} \ 2x'_{31}y'^3_{31} \ y'^4_{31} \ 0 \right) + \\
 n_{31,y} \left( 0 \ 0 \ 1 \ 0 \ x'_{31} \ 2y'_{31} \ 0 \ x'^2_{31} \ 2x'_{31}y'_{31} \ 3y'^2_{31} \ 0 \ x'^3_{31} \ 2x'^2_{31}y'_{31} \ 3x'_{31}y'^2_{31} \ 4y'^3_{31} \ 0 \ x'^4_{31} \ 2x'^3_{31}y'_{31} \ 3x'^2_{31}y'^2_{31} \ 4x'_{31}y'^3_{31} \ 5y'^4_{31} \right)
 \end{array} \right] \mathbf{c} = \left\{ \begin{array}{l} L\varphi_{,n}^{12} \\ L\varphi_{,n}^{23} \\ L\varphi_{,n}^{31} \end{array} \right\}$$

Matricialmente, el proceso realizado puede escribirse:

$$\mathbf{A}\mathbf{c} = \mathbf{L}\Phi \quad \rightarrow \quad \mathbf{c} = \mathbf{A}^{-1}\mathbf{L}\Phi$$

$$\Phi^T = \left[ \varphi^1 \ \varphi^1_{,x} \ \varphi^1_{,y} \ \varphi^1_{,xx} \ \varphi^1_{,xy} \ \varphi^1_{,yy} \ \varphi^2 \ \varphi^2_{,x} \ \varphi^2_{,y} \ \varphi^2_{,xx} \ \varphi^2_{,xy} \ \varphi^2_{,yy} \ \varphi^3 \ \varphi^3_{,x} \ \varphi^3_{,y} \ \varphi^3_{,xx} \ \varphi^3_{,xy} \ \varphi^3_{,yy} \ \varphi_n^{12} \ \varphi_n^{23} \ \varphi_n^{31} \right]$$

$$\mathbf{L} = \text{diag} [1 \ L \ L \ L^2 \ L^2 \ L^2 \ 1 \ L \ L \ L^2 \ L^2 \ L^2 \ 1 \ L \ L \ L^2 \ L^2 \ L^2 \ L \ L \ L]$$

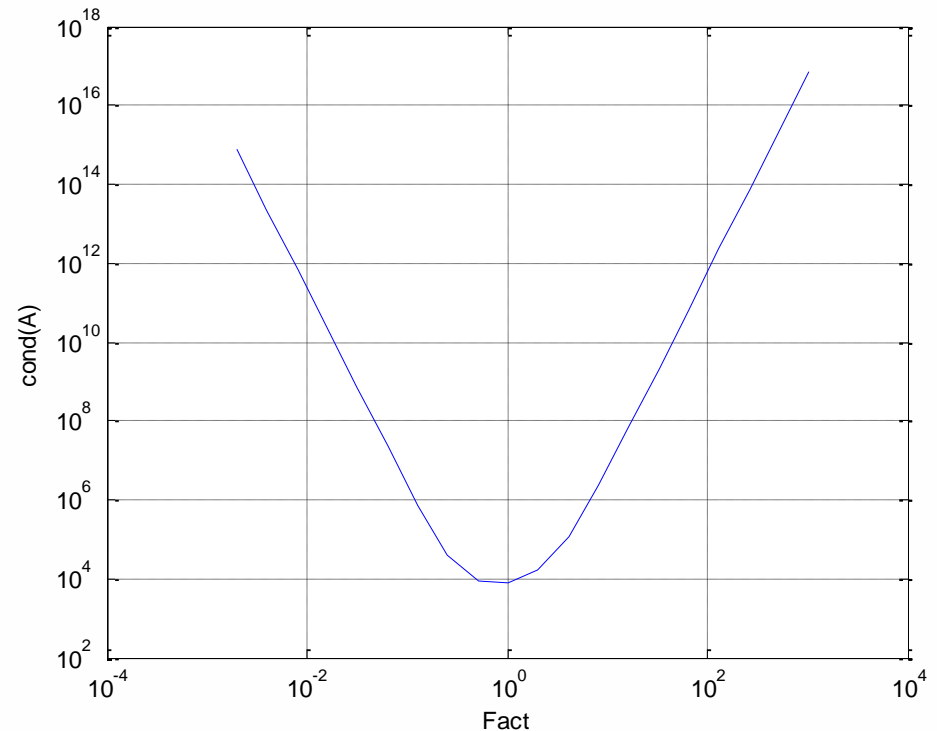
$$\mathbf{c} = \left[ c_{00} \ c_{10} \ c_{01} \ c_{20} \ c_{11} \ c_{02} \ c_{30} \ c_{21} \ c_{12} \ c_{03} \ c_{40} \ c_{31} \ c_{22} \ c_{13} \ \cdots \ c_{23} \ c_{14} \ c_{05} \right]^T$$

## Elemento P5 - Desarrollo del MEF para Stokes

El parámetro de escala  $L$  se elige de forma de condicionar adecuadamente la matriz  $\mathbf{A}$  y así evitar problemas al invertirla.

$$L = \frac{1}{2} \max \left( \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}, \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} \right)$$

La figura muestra la variación de la condición de  $\mathbf{A}$  para un triángulo arbitrario (coordenadas aleatorias) en función de un parámetro de escala igual a  $Fact * L$



# Elemento P5 - Desarrollo del MEF para Stokes

$$\mu \int_{\Omega} \nabla v_i \cdot \nabla u_i \, dx = \int_{\Omega} v_i f_i \, dx$$

$$\mathbf{u} = \begin{Bmatrix} \frac{\partial \varphi}{\partial y} \\ -\frac{\partial \varphi}{\partial x} \end{Bmatrix} \rightarrow$$

$$\nabla u_1 = \begin{Bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_1}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 \varphi}{\partial x \partial y} \\ \frac{\partial^2 \varphi}{\partial y^2} \end{Bmatrix}$$

$$\nabla u_2 = \begin{Bmatrix} \frac{\partial u_2}{\partial x} \\ \frac{\partial u_2}{\partial y} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 \varphi}{\partial x^2} \\ -\frac{\partial^2 \varphi}{\partial x \partial y} \end{Bmatrix}$$

$$\varphi(x, y) = \sum_{0 \leq i+j \leq 5} c_{ij} x'^i y'^j = c_{00} + c_{10}x' + c_{01}y' + c_{20}x'^2 + c_{11}x'y' + c_{02}y'^2 + \dots = \mathbf{p}^T(x, y) \mathbf{c}, \quad \forall (x, y) \in \mathbf{K}.$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \sum \frac{i}{L} \frac{i-1}{L} c_{ij} x'^{(i-2)} y'^j = \frac{2}{L^2} c_{20} + \frac{6}{L^2} c_{30}x' + \frac{2}{L^2} c_{21}x'^2 y' + \dots =$$

$$= \left[ 0 \ 0 \ 0 \ \frac{2}{L^2} \ 0 \ 0 \ \frac{6}{L^2} x' \ \frac{2}{L^2} y' \ 0 \ 0 \ \frac{12}{L^2} x'^2 \ \frac{6}{L^2} x' y' \ \frac{2}{L^2} y'^2 \ 0 \ 0 \ \frac{20}{L^2} x'^3 \ \frac{12}{L^2} x'^2 y' \ \frac{6}{L^2} x' y'^2 \ \frac{2}{L^2} y'^3 \ 0 \ 0 \right] \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right)^T \mathbf{c}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \left[ 0 \ 0 \ 0 \ 0 \ \frac{1}{L^2} \ 0 \ 0 \ \frac{2}{L^2} x' \ \frac{2}{L^2} y' \ 0 \ 0 \ \frac{3}{L^2} x'^2 \ \frac{4}{L^2} x' y' \ \frac{3}{L^2} y'^2 \ 0 \ 0 \ \frac{4}{L^2} x'^3 \ \frac{6}{L^2} x'^2 y' \ \frac{6}{L^2} x' y'^2 \ \frac{4}{L^2} y'^3 \ 0 \right] \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right)^T \mathbf{c}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \left[ 0 \ 0 \ 0 \ 0 \ 0 \ \frac{2}{L^2} \ 0 \ 0 \ \frac{2}{L^2} x' \ \frac{6}{L^2} y' \ 0 \ 0 \ \frac{2}{L^2} x'^2 \ \frac{6}{L^2} x' y' \ \frac{12}{L^2} y'^2 \ 0 \ 0 \ \frac{2}{L^2} x'^3 \ \frac{6}{L^2} x'^2 y' \ \frac{12}{L^2} x' y'^2 \ \frac{20}{L^2} y'^3 \right] \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right)^T \mathbf{c}$$

## Elemento P5 - Desarrollo del MEF para Stokes

$$\mu \int_{\Omega} \nabla v_i \cdot \nabla u_i \, dx = \mu \int_{\Omega} (\nabla v_1 \cdot \nabla u_1 + \nabla v_2 \cdot \nabla u_2) \, dx = \mu \int_{\Omega} \left( \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \phi}{\partial y^2} \right) dx \quad \forall \psi \in W_h$$

Usando:

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} &= \mathbf{c}^T \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right) = \boldsymbol{\Psi}^T \mathbf{L} \mathbf{A}^{-T} \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right) & \frac{\partial^2 \phi}{\partial x^2} &= \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right)^T \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right)^T \mathbf{A}^{-1} \mathbf{L} \boldsymbol{\Phi} \\ \frac{\partial^2 \psi}{\partial x \partial y} &= \mathbf{c}^T \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right) = \boldsymbol{\Psi}^T \mathbf{L} \mathbf{A}^{-T} \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right) & \frac{\partial^2 \phi}{\partial x \partial y} &= \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right)^T \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right)^T \mathbf{A}^{-1} \mathbf{L} \boldsymbol{\Phi} \\ \frac{\partial^2 \psi}{\partial y^2} &= \mathbf{c}^T \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right) = \boldsymbol{\Psi}^T \mathbf{L} \mathbf{A}^{-T} \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right) & \frac{\partial^2 \phi}{\partial y^2} &= \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right)^T \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right)^T \mathbf{A}^{-1} \mathbf{L} \boldsymbol{\Phi} \end{aligned}$$

y teniendo en cuenta que los parámetros  $\boldsymbol{\Psi}$  son arbitrarios, obtenemos:

$$\mathbf{K} = \mu \mathbf{L} \mathbf{A}^{-T} \int_{\Omega} \underbrace{\left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \frac{\partial^2 \mathbf{p}^T}{\partial x^2} + 2 \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \frac{\partial^2 \mathbf{p}^T}{\partial x \partial y} + \frac{\partial^2 \mathbf{p}}{\partial y^2} \frac{\partial^2 \mathbf{p}^T}{\partial y^2} \right) dx}_{\text{no nulos últimos } 18 \times 18 \text{ (primeros tres términos, constante, lineal en } x, y, \text{ nulos)}} \mathbf{A}^{-1} \mathbf{L} = \mu \mathbf{L} \mathbf{A}^{-T} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \\ & & 18 \times 18 \end{bmatrix} \mathbf{A}^{-1} \mathbf{L}$$

## Elemento P5 - Desarrollo del MEF para Stokes

Si introducimos la hipótesis cinemática siguiente (cálculo de derivadas en el punto medio a partir de las derivadas en los nodos vértice):

$$\begin{Bmatrix} \varphi_{,n}^{12} \\ \varphi_{,n}^{23} \\ \varphi_{,n}^{31} \end{Bmatrix} = \begin{Bmatrix} \frac{n_{12x}}{2} \varphi_{,x}^1 + \frac{n_{12y}}{2} \varphi_{,y}^1 + \frac{n_{12x}}{2} \varphi_{,x}^2 + \frac{n_{12y}}{2} \varphi_{,y}^2 \\ \frac{n_{23x}}{2} \varphi_{,x}^2 + \frac{n_{23y}}{2} \varphi_{,y}^2 + \frac{n_{23x}}{2} \varphi_{,x}^3 + \frac{n_{23y}}{2} \varphi_{,y}^3 \\ \frac{n_{31x}}{2} \varphi_{,x}^3 + \frac{n_{31y}}{2} \varphi_{,y}^3 + \frac{n_{31x}}{2} \varphi_{,x}^1 + \frac{n_{31y}}{2} \varphi_{,y}^1 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & n_{12x} & n_{12x} & 0 & n_{12y} & n_{12y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_{23x} & n_{23x} & 0 & n_{23y} & n_{23y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_{31x} & 0 & n_{31x} & n_{31y} & 0 & n_{31y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Phi_{1:18}$$

$$\begin{Bmatrix} \varphi_{,n}^{12} \\ \varphi_{,n}^{23} \\ \varphi_{,n}^{31} \end{Bmatrix} = \mathbf{D} \Phi_{1:18}$$

podemos luego expresar la matriz de rigidez de 18x18 (con grados de libertad sólo en los vértices) como:

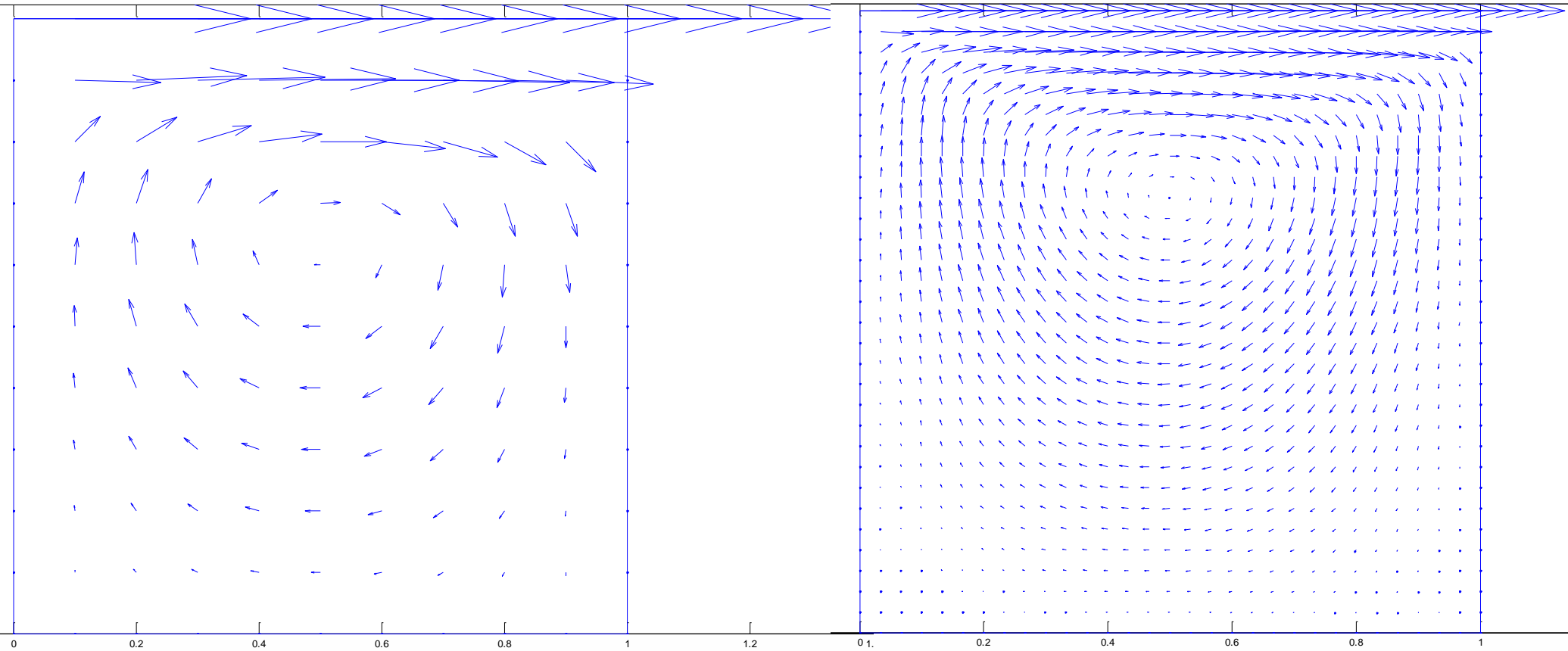
$$\mathbf{K} = \mu \mathbf{B}^T \mathbf{P} \mathbf{B}$$

siendo:

$$\mathbf{B}_{18 \times 18} \triangleq \begin{pmatrix} \mathbf{A}^{-1} \end{pmatrix}_{(4:21,1:21)} \begin{bmatrix} \mathbf{I}_{18 \times 18} \\ \mathbf{D}_{3 \times 18} \end{bmatrix}$$

# Ejemplo: Flujo en una Cavity Cuadrada

## 1) Elemento sin escalado ni traslacion



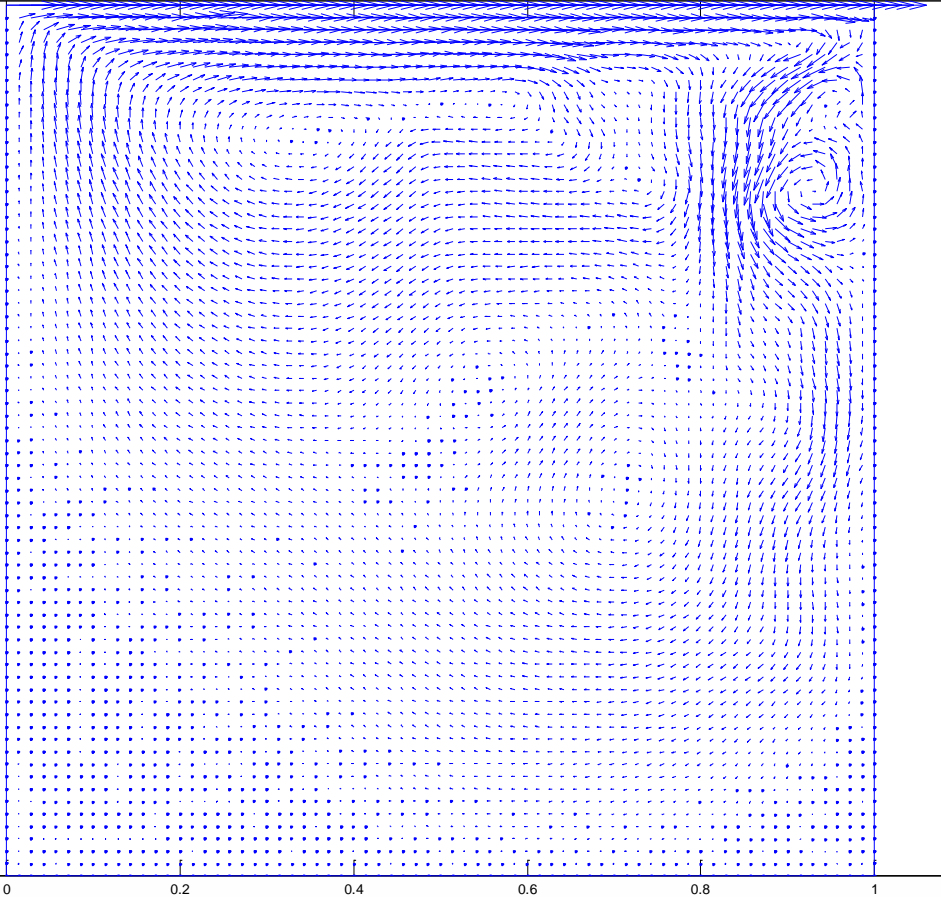
Malla 10x10x2 triángulos

Malla 30x30x2 triángulos

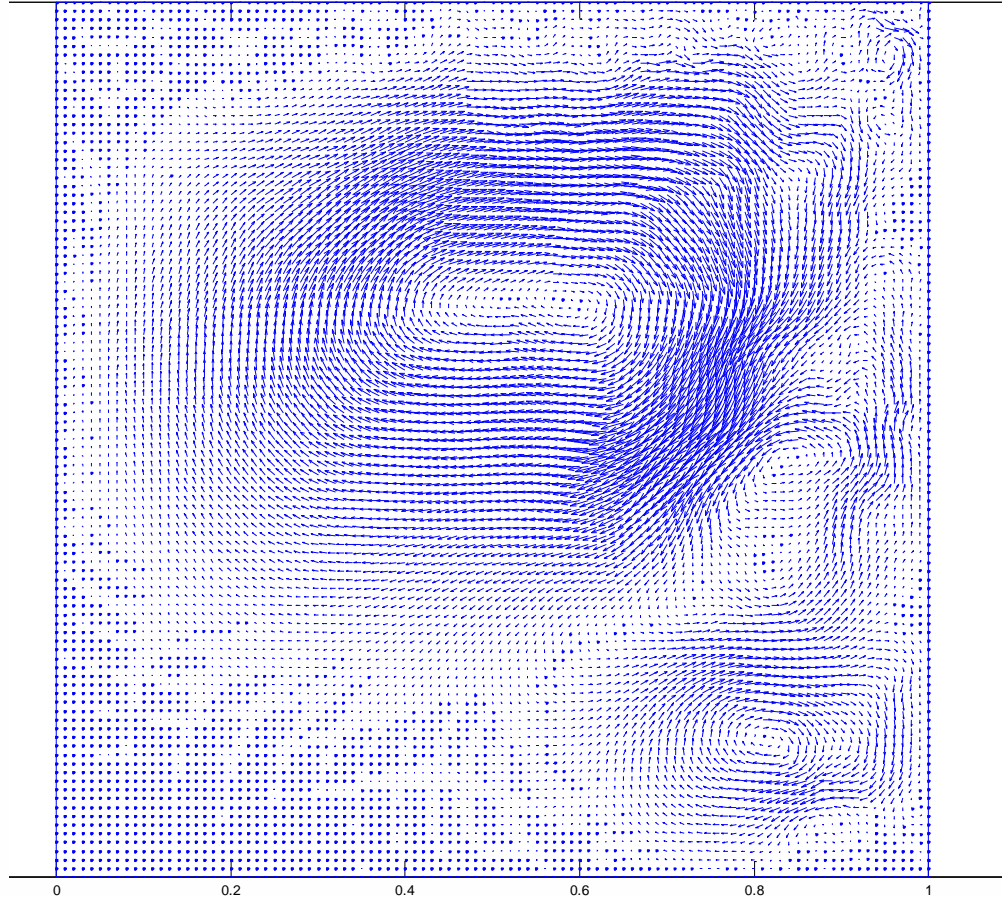


# Ejemplo: Flujo en una Cavidad Cuadrada

## 1) Elemento sin escalado ni traslación



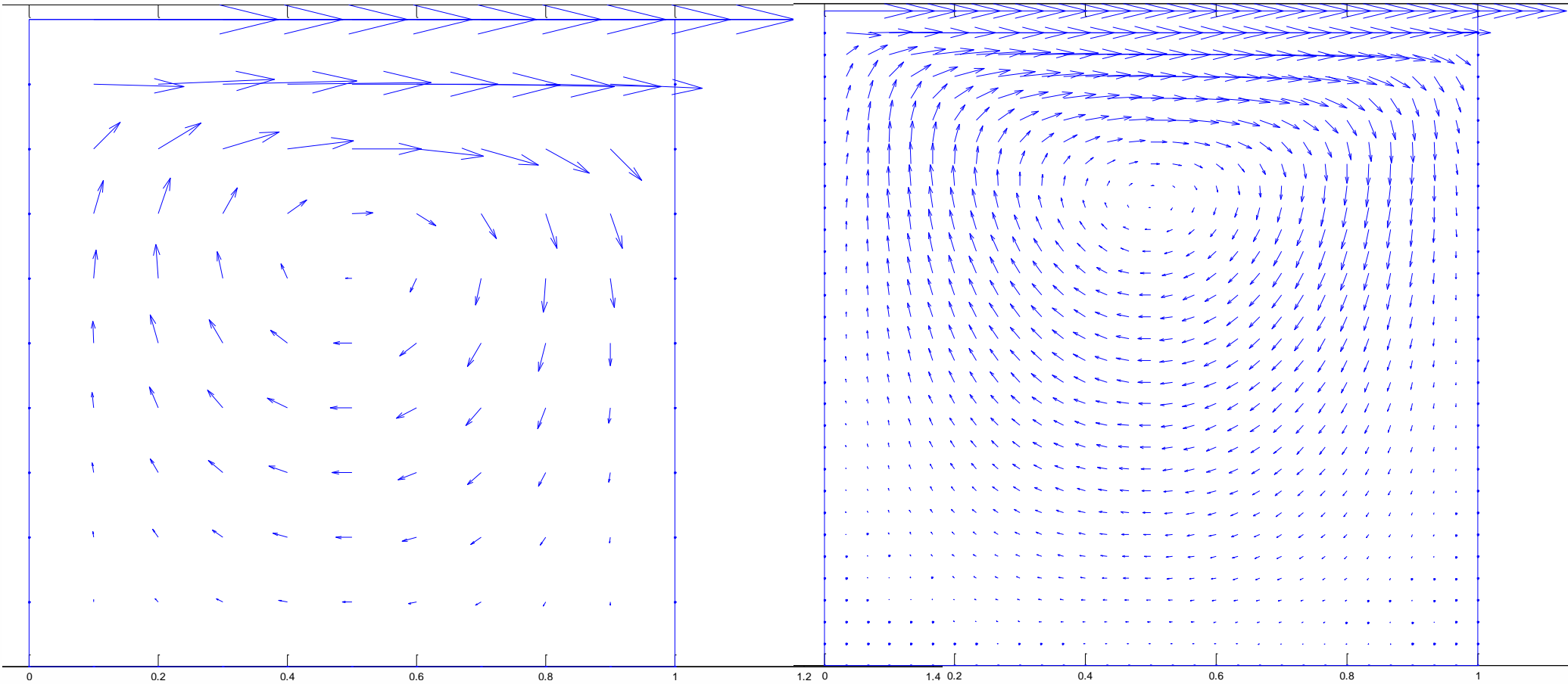
Malla 70x70x2 triángulos



Malla 100x100x2 triángulos

# Ejemplo: Flujo en una Cavity Cuadrada

## 2) Elemento c/escalado

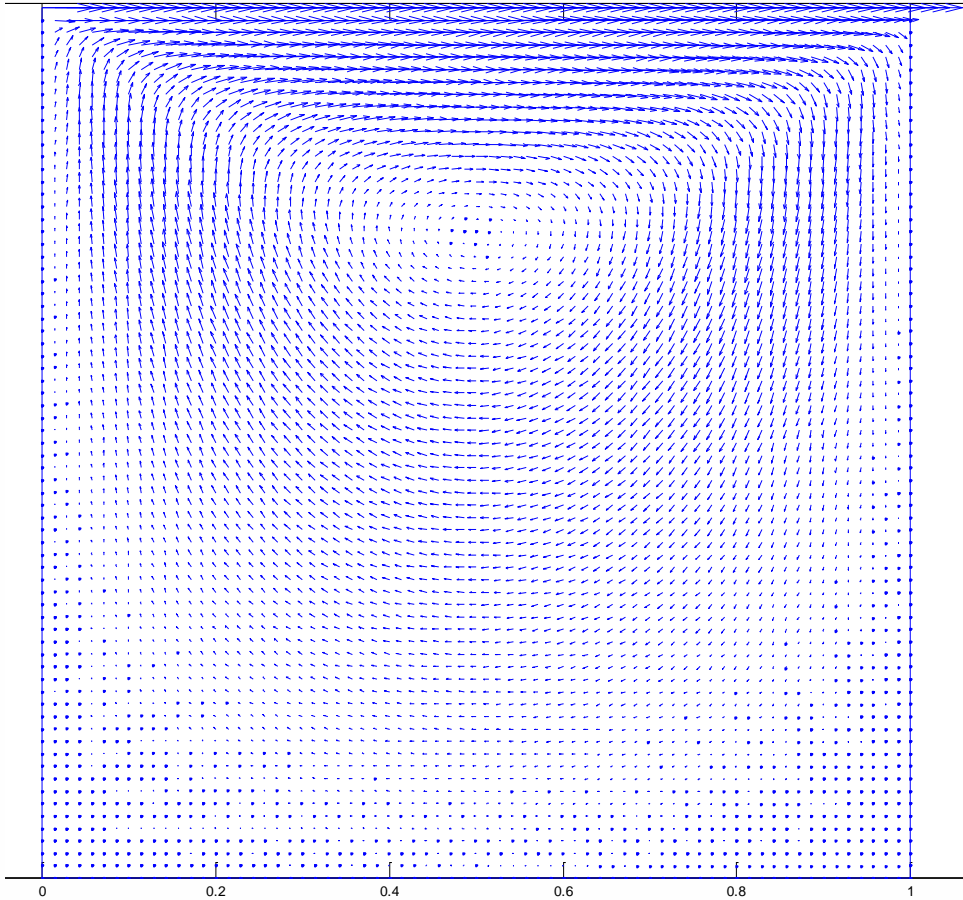


Malla 10x10x2 triángulos

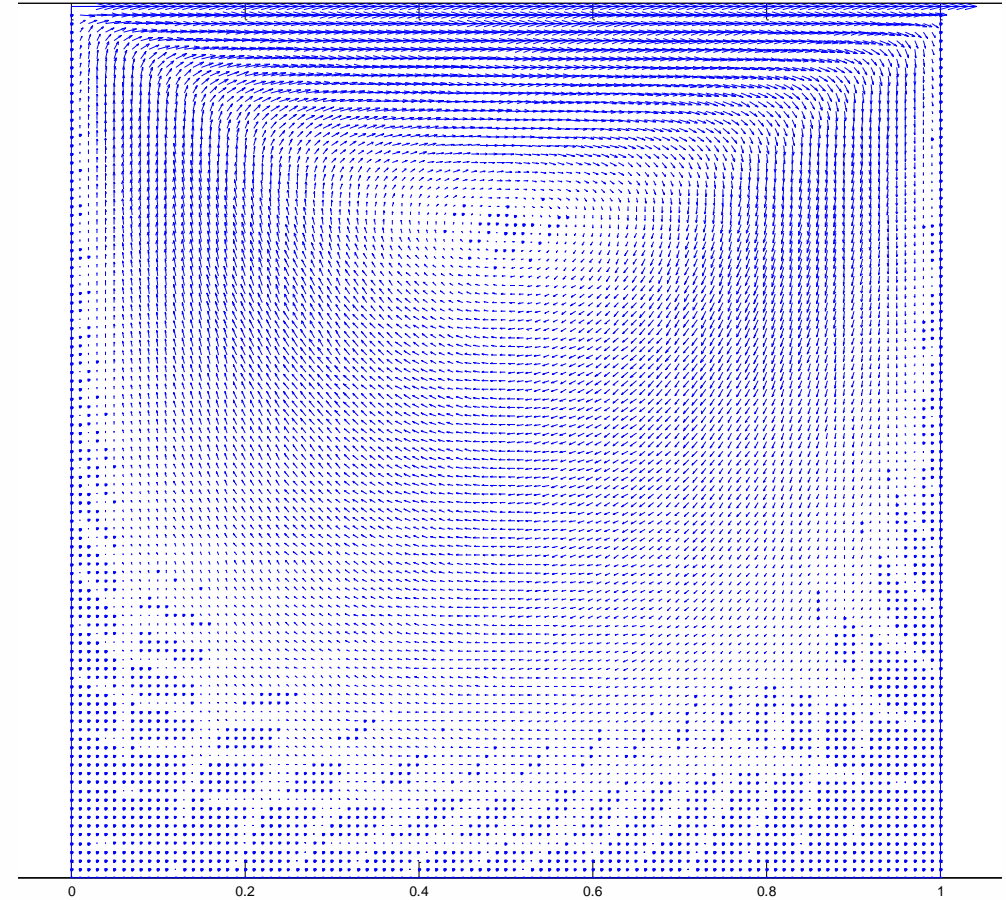
Malla 30x30x2 triángulos

# Ejemplo: Flujo en una Cavity Cuadrada

## 2) Elemento c/escalado



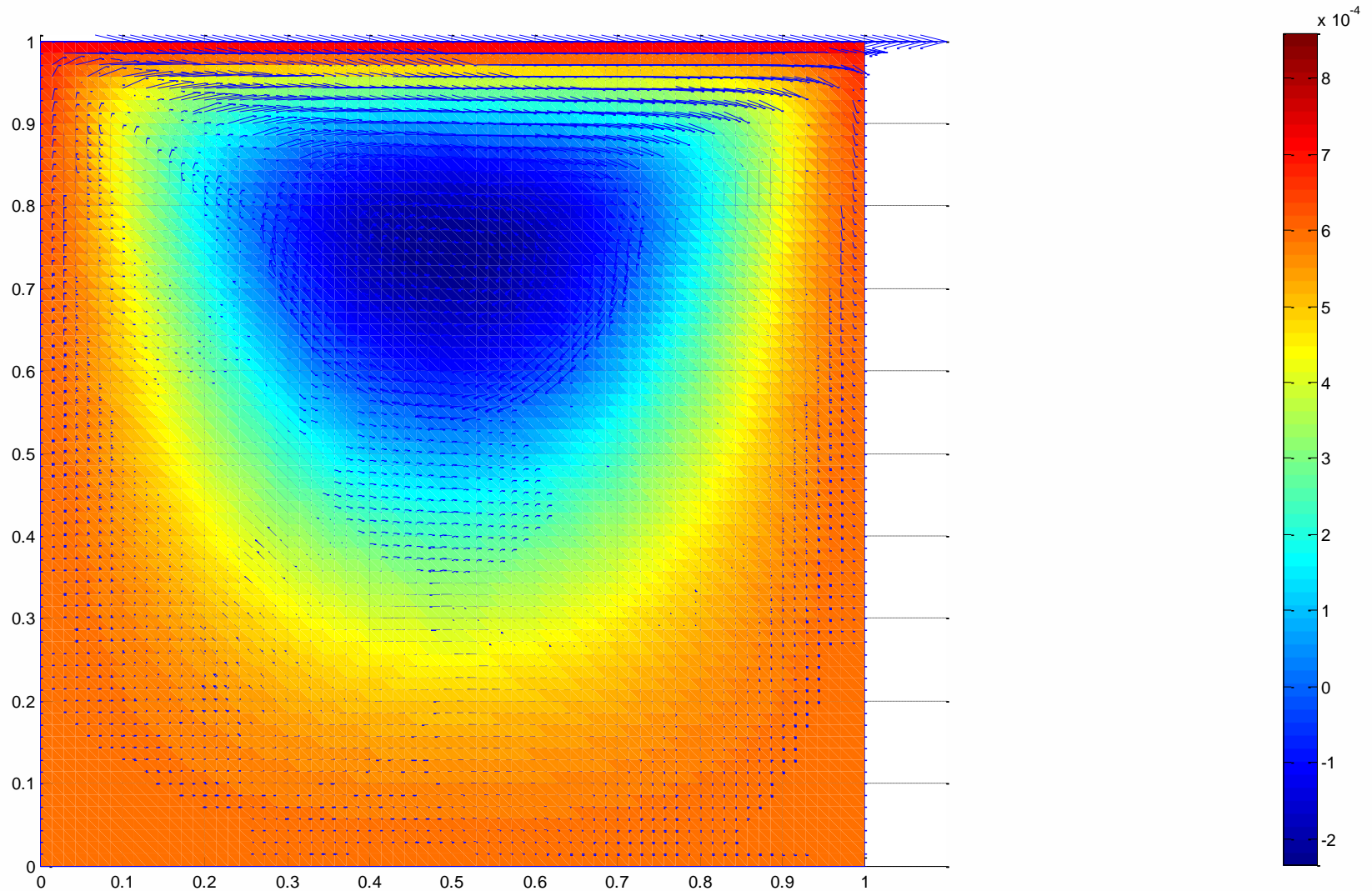
Malla 70x70x2 triángulos



Malla 100x100x2 triángulos

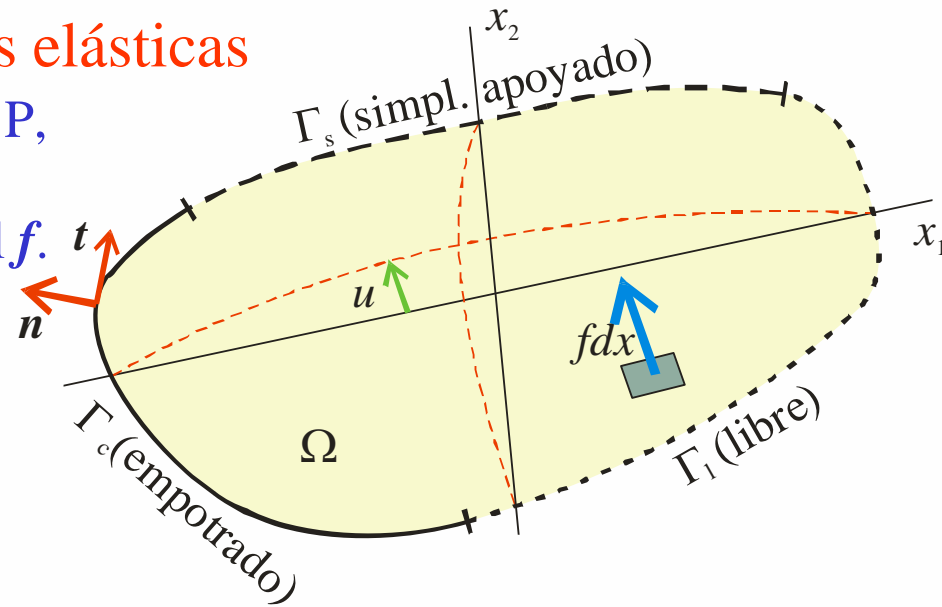
# Ejemplo: Flujo en una Cavity Cuadrada

Función línea de corriente (70 x 70)



# Flexión de placas elásticas

- Consideremos una delgada placa elástica  $P$ , cuya superficie media está dada por el dominio  $\Omega \subset \mathbb{R}^2$ , sujeta a carga transversal  $f$ .
- Se desea determinar
  - deflexión transversal  $u$
  - momentos  $M_{ij}$ ,  $i, j=1, 2$ .
- Los momentos están definidos:



$$M_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} z \, dz$$

$$M_{11}, M_{22}$$

Momentos flectores

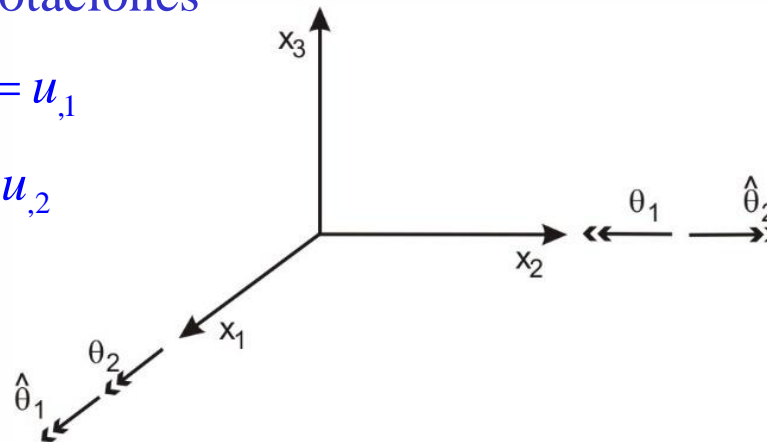
$$M_{12}, M_{21}$$

Momentos de “twist”

- Convención de signos para rotaciones

$$\theta_1 = -\hat{\theta}_2 = u_{,1}$$

$$\theta_2 = \hat{\theta}_1 = u_{,2}$$



## Flexión de placas elásticas (cont.)

- El problema está gobernado por

$$M_{ij,ij} = f \quad \text{en } \Omega \quad \text{Ecuación de equilibrio}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{sobre } \Gamma_c \quad \text{CB empotrado}$$

$$u = M_{nn} = 0 \quad \text{sobre } \Gamma_s \quad \text{CB simpl. apoyado}$$

$$M_{nn} = Q(\mathbf{M}) = 0 \quad \text{sobre } \Gamma_l \quad \text{CB libre}$$

$$Q(\mathbf{M}) = M_{ij,j}n_i + \frac{\partial M_{nt}}{\partial t} \quad \text{Fuerza de corte o transversal}$$

## Flexión de placas elásticas (cont.)

- Ecuaciones de clausura
  - Asumiendo pequeñas deflexiones y material elástico lineal, la ecuación constitutiva (ley de Hooke) toma la forma

$$M_{ij} = \lambda \Delta u \delta_{ij} + \mu \chi_{ij} \quad \left| \begin{array}{l} \lambda, \mu \in \mathbb{R}^+ \quad \text{constantes} \\ \chi_{ij} = u_{,ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{curvatura} \end{array} \right.$$

- Las constantes  $\lambda$  y  $\mu$  dependen del módulo de elasticidad  $E$  y del coef. de Poisson  $\nu$ , así como del espesor de la placa  $h$ , de acuerdo a

$$\lambda = \frac{Eh^3}{12(1+\nu)} \quad \mu = \frac{\nu Eh^3}{12(1-\nu^2)}$$

# Formulación variacional del problema de flexión de placas elásticas

1. Adoptamos el espacio  $V = \left\{ v : v \in H^2(\Omega), v = \frac{\partial v}{\partial n} = 0 \text{ en } \Gamma_c, v = 0 \text{ en } \Gamma_s \right\}$

2. Hacemos

$$\int_{\Omega} f v dx = \int_{\Omega} M_{ij,ij} v dx$$

$$\int_{\Omega} f v dx = - \int_{\Omega} M_{ij,j} v_{,i} dx + \int_{\Gamma} M_{ij,j} n_i v ds$$

$$\int_{\Omega} f v dx = \int_{\Omega} M_{ij} v_{,ij} dx - \int_{\Gamma} \overbrace{M_{ij} n_j v_{,i}} ds + \int_{\Gamma} M_{ij,j} n_i v ds$$

$$v_{,i} = \frac{\partial v}{\partial n} n_i + \frac{\partial v}{\partial t} t_i$$

$$M_{ij} n_j v_{,i} = M_{ij} n_i n_j \frac{\partial v}{\partial n} + M_{ij} t_i n_j \frac{\partial v}{\partial t}$$

$$= M_{nn} \frac{\partial v}{\partial n} + M_{nt} \frac{\partial v}{\partial t}$$

$$\int_{\Omega} f v dx = \int_{\Omega} M_{ij} \chi_{ij}(v) dx - \int_{\Gamma} M_{nn} \frac{\partial v}{\partial n} ds - \int_{\Gamma} M_{nt} \frac{\partial v}{\partial t} ds + \int_{\Gamma} M_{ij,j} n_i v ds$$

$$\int_{\Omega} f v dx = \int_{\Omega} M_{ij} \chi_{ij}(v) dx - \int_{\Gamma} M_{nn} \frac{\partial v}{\partial n} ds + \int_{\Gamma} \frac{\partial M_{nt}}{\partial t} v ds + \int_{\Gamma} M_{ij,j} n_i v ds$$

Integración por partes (con  $\Gamma$  suave)

$$\int_{\Omega} f v dx = \int_{\Omega} M_{ij} \chi_{ij}(v) dx - \underbrace{\int_{\Gamma} M_{nn} \frac{\partial v}{\partial n} ds}_{=0} + \underbrace{\int_{\Gamma} \left( M_{ij,j} n_i + \frac{\partial M_{nt}}{\partial t} \right) v ds}_{=0}$$

$$\int_{\Omega} f v dx = \int_{\Omega} (\lambda \Delta u \delta_{ij} + \mu \chi_{ij}) \chi_{ij}(v) dx = \int_{\Omega} [\lambda \Delta u \Delta v + \mu \chi_{ij}(u) \chi_{ij}(v)] dx$$

$L(v)$

$a(u, v)$



# Formulación variacional del problema de flexión de placas elásticas (cont.)

- La forma variacional del problema de flexión de placas elásticas resulta

$$(V) \quad \text{Hallar } u \in V / a(u, v) = L(v), \quad \forall v \in V$$

$$\text{con: } a(u, v) = \int_{\Omega} \left[ \lambda \Delta u \Delta v + \mu \chi_{ij}(u) \chi_{ij}(v) \right] dx$$

$$L(v) = \int_{\Omega} f v dx$$

$$V = \left\{ v : v \in H^2(\Omega), v = \frac{\partial v}{\partial n} = 0 \text{ en } \Gamma_c, v = 0 \text{ en } \Gamma_s \right\}$$

- La forma bilineal  $a(.,.)$  es en general simétrica y continua.
- Además,  $a(.,.)$  es V-elíptica si  $\Gamma_c > 0$ , i.e., si la placa está empotrada a lo largo de una parte de su borde.
- La forma lineal  $L(.)$  es continua.

# Formulación variacional del problema de flexión de placas elásticas (cont.)

- Definiendo:

$$\boldsymbol{\kappa}(u) = \left\{ \begin{array}{l} \frac{\partial^2 u}{\partial x_1^2} \\ \frac{\partial^2 u}{\partial x_2^2} \\ 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \end{array} \right\} \quad \mathbf{D} = \begin{bmatrix} \lambda + \mu & \lambda & 0 \\ \lambda & \lambda + \mu & 0 \\ 0 & 0 & \mu/2 \end{bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & 1-\nu & 0 \\ 1-\nu & 1 & 0 \\ 0 & 0 & \nu/2 \end{bmatrix}$$

expresamos la forma bilineal matricialmente:

$$a(u, v) = \int_{\Omega} \boldsymbol{\kappa}(u) \cdot \mathbf{D} \boldsymbol{\kappa}(v) \, dx$$

- Ahora se puede formular el MEF para el problema de flexión de placas elásticas usando el elemento  $C^1$ -continuo ya descrito.

## MEF no conforme aplicado al problema de Stokes

Consideremos el problema de Stokes en  $\Omega \subset \mathbb{R}^2$ . Luego:

$$\mathbf{V} = \left\{ \mathbf{v} : \mathbf{v} = (v_1, v_2) \in \left[ \mathbf{H}_0^1(\Omega) \right]^2 \text{ y } \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \text{ en } \Omega \right\}$$

- Si  $\Omega$  es simplemente conexo (i.e., no contiene agujeros),  $\text{div } \mathbf{v} = 0$  en  $\Omega$  si y solo si

$$\mathbf{v} = \left( \frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right) = \text{rot } \varphi \text{ para alguna función } \varphi.$$

$\varphi$ : función de corriente del campo de velocidades  $\mathbf{v}$ .

o sea:  $\mathbf{v} \in \mathbf{V} \Leftrightarrow \mathbf{v} = \text{rot } \varphi, \varphi \in \mathbf{H}_0^2(\Omega)$ .

- Anteriormente, usamos un subespacio  $W_h$  de dimensión finita de  $\mathbf{H}_0^2(\Omega)$  (usamos el elemento finito  $C^1$ -continuo) y definimos  $\mathbf{V}_h = \{ \mathbf{v} : \mathbf{v} = \text{rot } \varphi, \varphi \in W_h \}$ .
- Probaremos ahora un elemento cuadrático NO CONFORME (o sea, no verifica que  $\mathbf{V}_h \subset \mathbf{H}_0^2(\Omega)$  !!!!)

## Elemento de Morley - Desarrollo del MEF para Stokes

Consiste en un EF triangular con polinomios cuadráticos. El campo  $\varphi(x, y)$  será aproximado por

$$\varphi(x, y) = \sum_{0 \leq i+j \leq 2} c_{ij} x'^i y'^j = c_{00} + c_{10}x' + c_{01}y' + c_{20}x'^2 + c_{11}x'y' + c_{02}y'^2 = \mathbf{p}^T(x, y)\mathbf{c}, \quad \forall (x, y) \in \mathbf{K}.$$

con:  $x' = \frac{x - x_0}{L}; \quad y' = \frac{y - y_0}{L}; \quad L$  : longitud característica de elemento

$(x_0, y_0)$  : centroide del elemento

$$\mathbf{p}^T(x, y) = \begin{bmatrix} 1 & x' & y' & x'^2 & x'y' & y'^2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} c_{00} & c_{10} & c_{01} & c_{20} & c_{11} & c_{02} \end{bmatrix}^T$$

$$L = \frac{1}{2} \max \left( \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}, \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} \right)$$

Para el cálculo de los coeficientes  $\mathbf{c}$  expresamos primero el valor del campo en los nodos

$$\varphi(x_k, y_k) = \sum_{0 \leq i+j \leq 2} c_{ij} x_k^i y_k^j = c_{00} + c_{10}x_k + c_{01}y_k + c_{20}x_k^2 + c_{11}x_k y_k + c_{02}y_k^2 = \varphi^k, \quad k = 1, 2, 3.$$

Obtenemos así tres ecuaciones con 6 incógnitas:

$$\begin{bmatrix} 1 & x_k & y_k & x_k^2 & x_k y_k & y_k^2 \end{bmatrix} \mathbf{c} = \varphi^k \quad k = 1, 2, 3.$$

## Elemento de Morley - Desarrollo del MEF para Stokes

Las ecuaciones restantes se obtienen calculando la derivada normal en los puntos medios de los lados:

$$\begin{bmatrix} n_{12x}(0 \ 1 \ 0 \ 2x'_{12} \ y'_{12} \ 0) + n_{12y}(0 \ 0 \ 1 \ 0 \ x'_{12} \ 2y'_{12}) \\ n_{23x}(0 \ 1 \ 0 \ 2x'_{23} \ y'_{23} \ 0) + n_{23y}(0 \ 0 \ 1 \ 0 \ x'_{23} \ 2y'_{23}) \\ n_{31x}(0 \ 1 \ 0 \ 2x'_{31} \ y'_{31} \ 0) + n_{31y}(0 \ 0 \ 1 \ 0 \ x'_{31} \ 2y'_{31}) \end{bmatrix} \mathbf{c} = \begin{Bmatrix} L\varphi_{,n}^{12} \\ L\varphi_{,n}^{23} \\ L\varphi_{,n}^{31} \end{Bmatrix}$$

Matricialmente, el proceso realizado puede escribirse:

$$\mathbf{Ac} = \mathbf{L}\Phi \quad \rightarrow \quad \mathbf{c} = \mathbf{A}^{-1}\mathbf{L}\Phi$$

$$\Phi^T = [\varphi^1 \quad \varphi^2 \quad \varphi^3 \quad \varphi_{,n}^{12} \quad \varphi_{,n}^{23} \quad \varphi_{,n}^{31}] \quad \mathbf{c} = [c_{00} \quad c_{10} \quad c_{01} \quad c_{20} \quad c_{11} \quad c_{02}]^T$$

$$\mathbf{L} = \text{diag} [1 \quad 1 \quad 1 \quad L \quad L \quad L]$$

$$\mu \int_{\Omega} \nabla v_i \cdot \nabla u_i \, dx = \int_{\Omega} v_i f_i \, dx$$

$$\mathbf{u} = \begin{Bmatrix} \frac{\partial \varphi}{\partial y} \\ -\frac{\partial \varphi}{\partial x} \end{Bmatrix} \rightarrow$$

$$\nabla u_1 = \begin{Bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_1}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 \varphi}{\partial x \partial y} \\ \frac{\partial^2 \varphi}{\partial y^2} \end{Bmatrix}$$

$$\nabla u_2 = \begin{Bmatrix} \frac{\partial u_2}{\partial x} \\ \frac{\partial u_2}{\partial y} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 \varphi}{\partial x^2} \\ -\frac{\partial^2 \varphi}{\partial x \partial y} \end{Bmatrix}$$

## Elemento de Morley - Desarrollo del MEF para Stokes

$$\varphi(x, y) = \sum_{0 \leq i+j \leq 2} c_{ij} x'^i y'^j = c_{00} + c_{10}x' + c_{01}y' + c_{20}x'^2 + c_{11}x'y' + c_{02}y'^2 = \mathbf{p}^T(x, y)\mathbf{c}, \quad \forall (x, y) \in K.$$

$$\mathbf{u} = \begin{Bmatrix} \frac{\partial \varphi}{\partial y} \\ -\frac{\partial \varphi}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \sum \frac{i}{L} c_{ij} x'^i y'^{(j-1)} \\ -\sum \frac{i}{L} c_{ij} x'^{(i-1)} y'^j \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{L} & 0 & \frac{x'}{L} & \frac{2y'}{L} \\ 0 & -\frac{1}{L} & 0 & -\frac{2x'}{L} & -\frac{y'}{L} & 0 \end{bmatrix}}_{\mathbf{G}(x', y')} \mathbf{c} = \mathbf{G}(x', y') \mathbf{A}^{-1} \mathbf{L} \Phi$$

Las velocidades se calculan en post-tratamiento (no son variables primales). Ejemplo: en el centro del triángulo

$$\frac{\partial^2 \varphi}{\partial x^2} = \sum \frac{i}{L} \frac{i-1}{L} c_{ij} x'^{(i-2)} y'^j = \begin{bmatrix} 0 & 0 & 0 & \frac{2}{L^2} & 0 & 0 \end{bmatrix} \mathbf{c} = \frac{2}{L^2} c_{20}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{L^2} & 0 \end{bmatrix} \mathbf{c} = \frac{1}{L^2} c_{11}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{2}{L^2} \end{bmatrix} \mathbf{c} = \frac{2}{L^2} c_{02}$$

$$\mu \int_{\Omega} \nabla v_i \cdot \nabla u_i \, dx = \mu \int_{\Omega} (\nabla v_1 \cdot \nabla u_1 + \nabla v_2 \cdot \nabla u_2) \, dx = \mu \int_{\Omega} \left( \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \varphi}{\partial y^2} \right) dx \quad \forall \psi \in \mathbf{W}_h$$

$$K \int \left( \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial y^2} \right) dx$$

$$\psi(x,y) = p^T(x,y) c = p^T A^{-1} \Phi$$

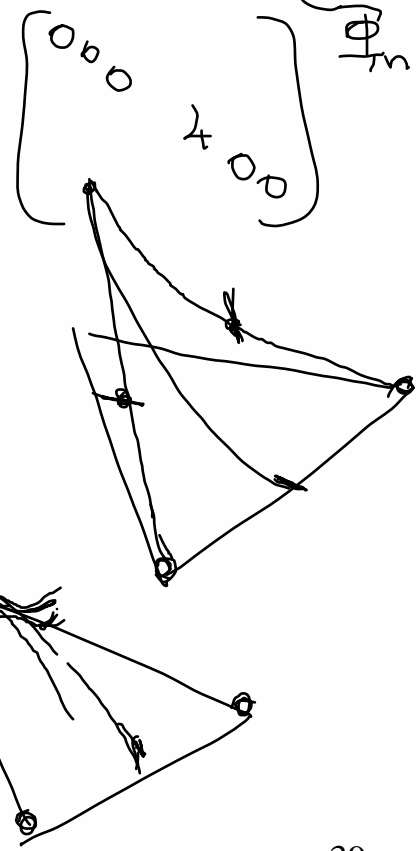
$$2c_2 = \frac{\partial^2 \psi}{\partial x^2} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} A^{-1} \Phi$$

$$\frac{\partial^2 \psi}{\partial x^2} = \psi^T A^{-T} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial x^2} = \psi^T A^{-T} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} A^{-1} \Phi$$

$$K = A^{-T} \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 2 \\ 0 & 4 & 2 & 4 \end{bmatrix} A^{-1} A_K$$

$$\Phi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 12 \\ 23 \\ 31 \end{matrix}$$



## Elemento de Morley - Desarrollo del MEF para Stokes

$$\mu \int_{\Omega} \nabla v_i \cdot \nabla u_i \, dx = \mu \int_{\Omega} (\nabla v_1 \cdot \nabla u_1 + \nabla v_2 \cdot \nabla u_2) \, dx = \mu \int_{\Omega} \left( \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \phi}{\partial y^2} \right) dx \quad \forall \psi \in W_h$$

Usando:

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} &= \mathbf{c}^T \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right) = \boldsymbol{\Psi}^T \mathbf{L} \mathbf{A}^{-T} \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right) & \frac{\partial^2 \phi}{\partial x^2} &= \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right)^T \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \right)^T \mathbf{A}^{-1} \mathbf{L} \boldsymbol{\Phi} \\ \frac{\partial^2 \psi}{\partial x \partial y} &= \mathbf{c}^T \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right) = \boldsymbol{\Psi}^T \mathbf{L} \mathbf{A}^{-T} \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right) & \frac{\partial^2 \phi}{\partial x \partial y} &= \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right)^T \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \right)^T \mathbf{A}^{-1} \mathbf{L} \boldsymbol{\Phi} \\ \frac{\partial^2 \psi}{\partial y^2} &= \mathbf{c}^T \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right) = \boldsymbol{\Psi}^T \mathbf{L} \mathbf{A}^{-T} \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right) & \frac{\partial^2 \phi}{\partial y^2} &= \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right)^T \mathbf{c} = \left( \frac{\partial^2 \mathbf{p}}{\partial y^2} \right)^T \mathbf{A}^{-1} \mathbf{L} \boldsymbol{\Phi} \end{aligned}$$

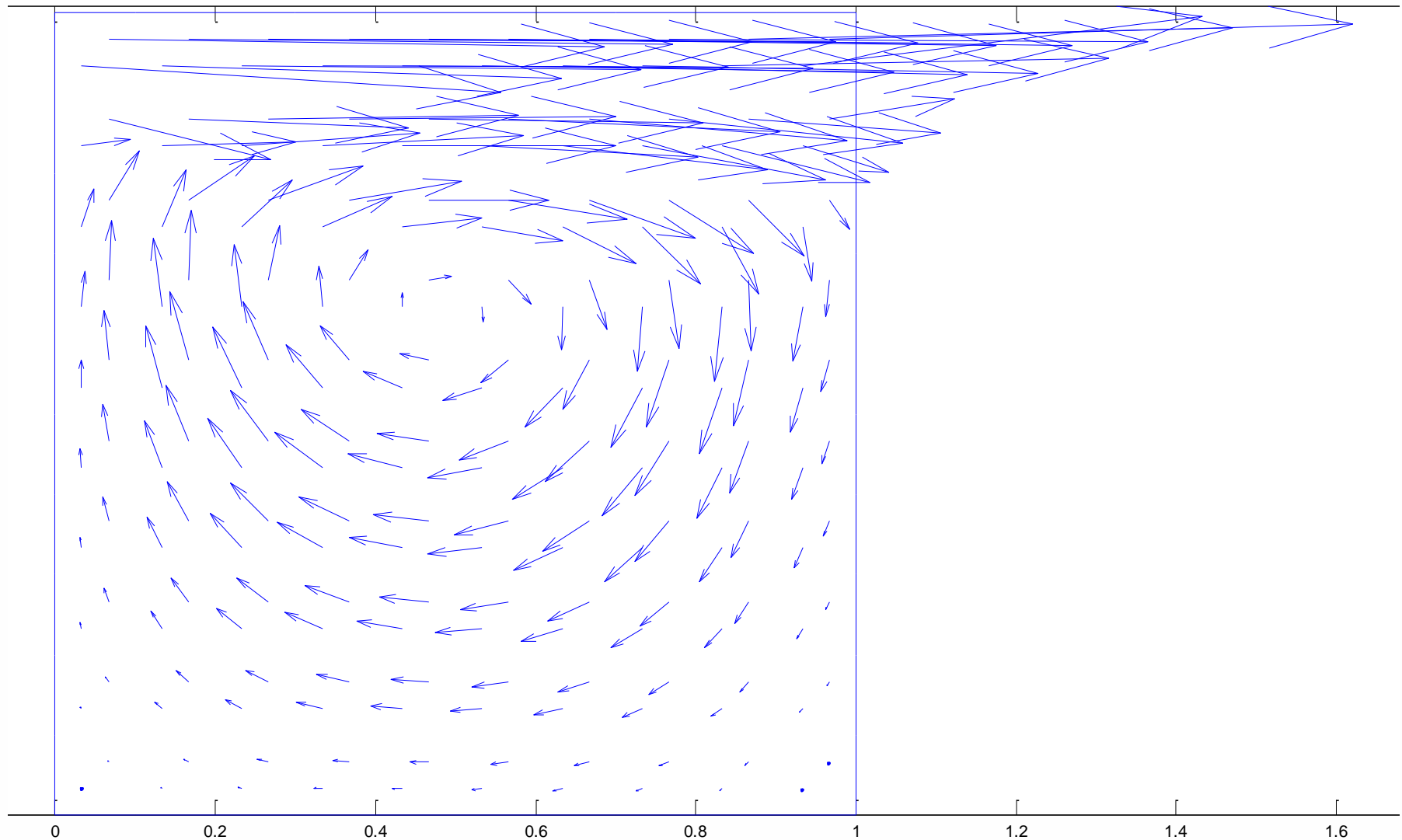
y teniendo en cuenta que los parámetros  $\boldsymbol{\Psi}$  son arbitrarios, obtenemos:

$$\mathbf{K} = \underbrace{\mu \mathbf{L} \mathbf{A}^{-T} \int_{\Omega} \left( \frac{\partial^2 \mathbf{p}}{\partial x^2} \frac{\partial^2 \mathbf{p}^T}{\partial x^2} + 2 \frac{\partial^2 \mathbf{p}}{\partial x \partial y} \frac{\partial^2 \mathbf{p}^T}{\partial x \partial y} + \frac{\partial^2 \mathbf{p}}{\partial y^2} \frac{\partial^2 \mathbf{p}^T}{\partial y^2} \right) dx}_{\text{no nulos últimos } 3 \times 3 \text{ términos}} \mathbf{A}^{-1} \mathbf{L} = \frac{\mu}{L^2} \mathbf{L} \mathbf{A}^{-T} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \end{bmatrix} \mathbf{A}^{-1} \mathbf{L} \mathbf{A}_K$$

donde  $\mathbf{A}_K$  es el área del elemento.

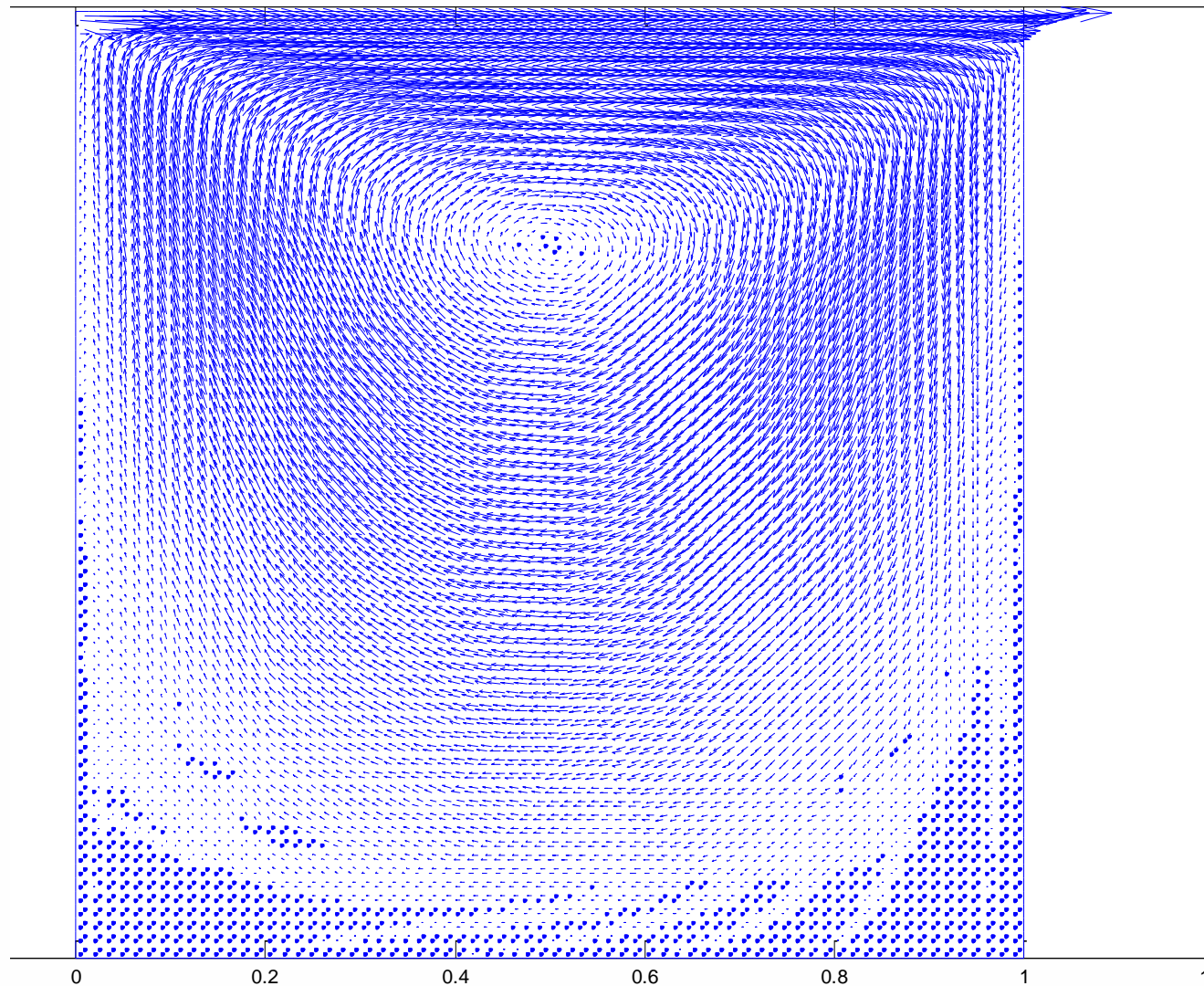


## Ejemplo elemento de Morley – Cavity cuadrada



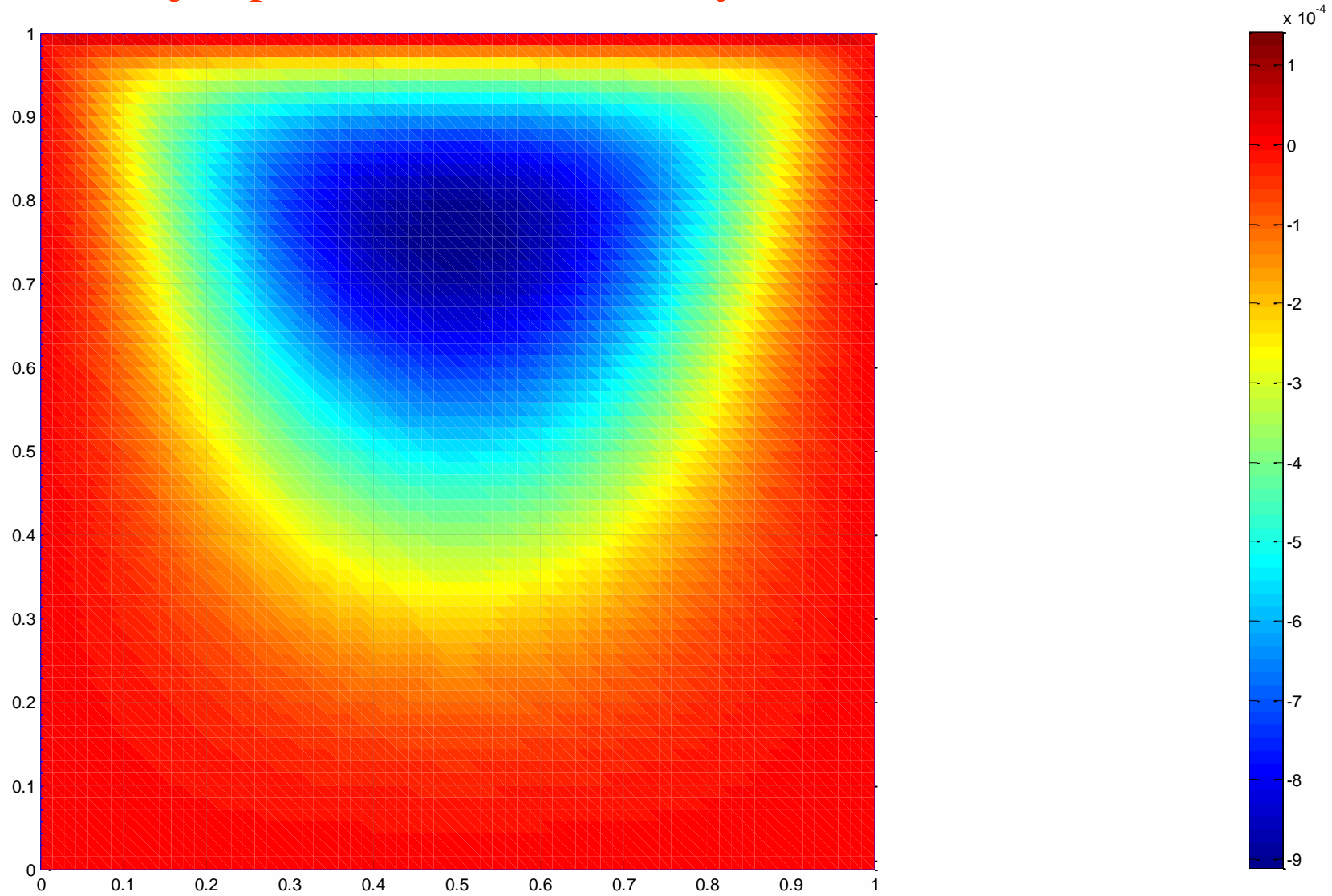
Malla 10x10x2 triángulos

## Ejemplo elemento de Morley – Cavity cuadrada



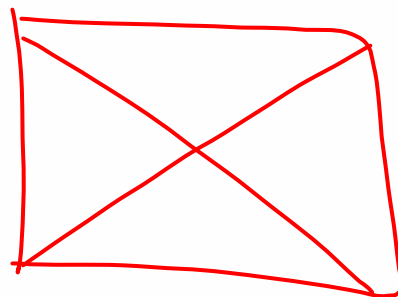
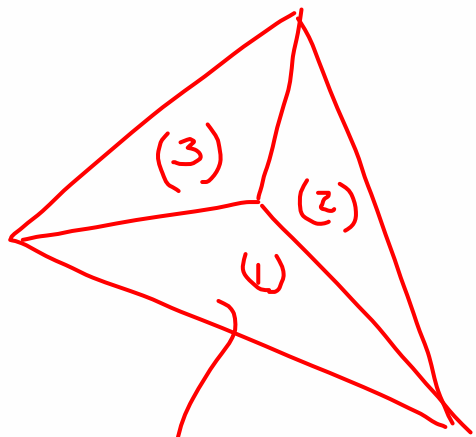
Malla 70x70x2 triángulos

# Ejemplo elemento de Morley – Cavity cuadrada



Malla 70x70x2 triángulos – Función línea de corriente

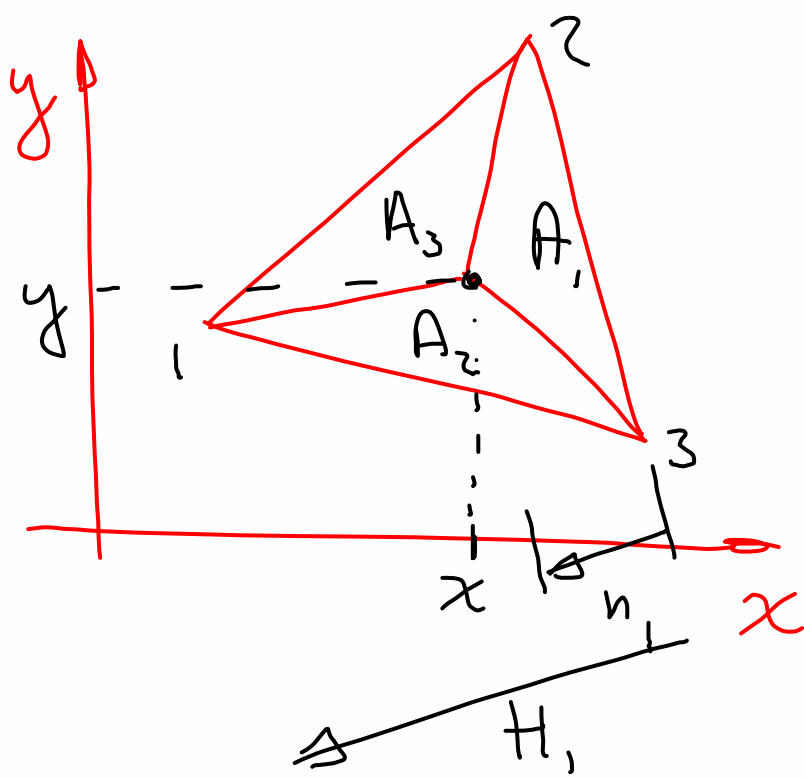
# "Patches" de elementos



$$u^{(e)} = \sum_{\pm} \phi_{\pm}^{(e)} U_{\pm}$$

LCCT-12 (Clough-Felipe)

Continuidad  $C^1$  y pol cúbicos.

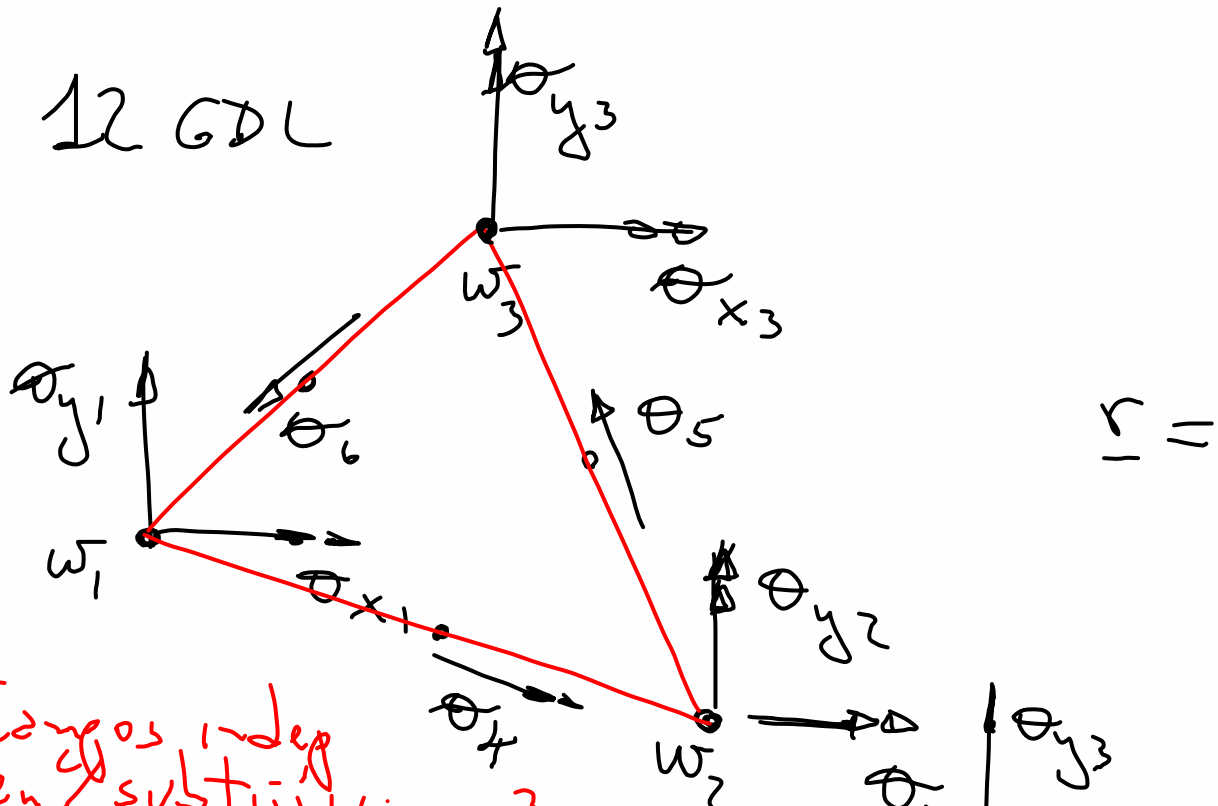


$\xi_1, \xi_2, \xi_3$  : coord axes

$$\xi_i = \frac{A_i}{A} = \frac{h_i}{h}$$

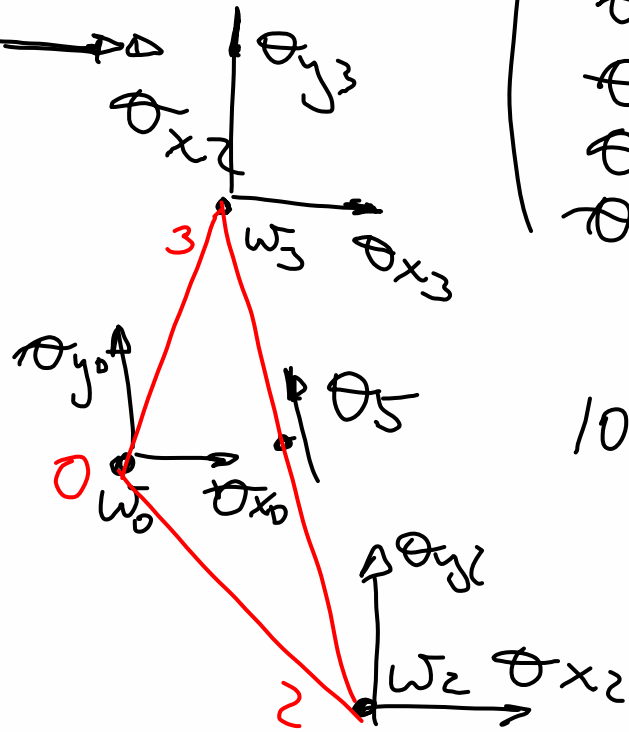
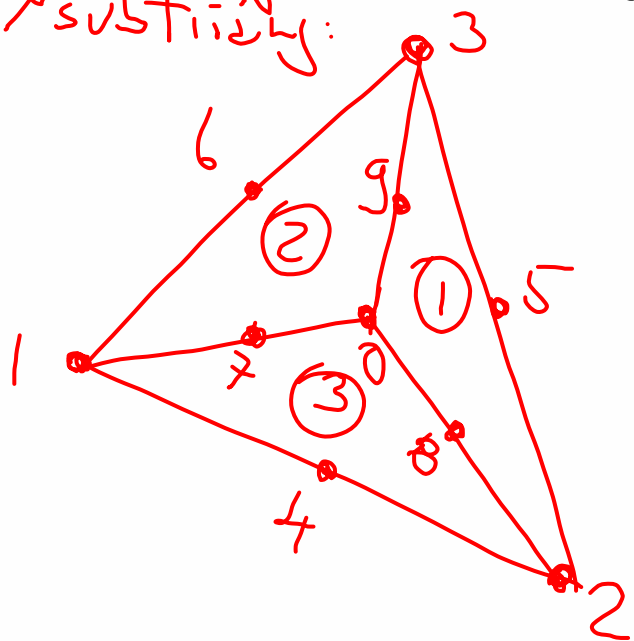
$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

12 GDL



- $v =$
- $w_1$
  - $\theta_{x_1}$
  - $\theta_{y_1}$
  - $w_2$
  - $\theta_{x_2}$
  - $\theta_{y_2}$
  - $w_3$
  - $\theta_{x_3}$
  - $\theta_{y_3}$
  - $\theta_4$
  - $\theta_5$
  - $\theta_6$

Campos indep en substitución:



10 GDL por cúbicos.

P/c sub elemento, expreso w como pol cúbico completo

$$\omega^{(i)} = \underline{\phi}^{(i)T} \underline{r}^{(i)} \quad i=1,2,3$$

Por ejemplo:

$$\underline{r}^{(i)T} = [w_2 \theta_{x_2} \theta_{y_2} \quad w_3 \theta_{x_3} \theta_{y_3} \quad w_0 \theta_{x_0} \theta_{y_0} \quad \theta_5]$$

Función de forma:

$$\underline{\phi}^{(i)} = \left( \begin{array}{c} \xi_1^2 (3 - 2\xi_1) + 6 \mu_3^{(i)} \xi_1 \xi_2 \xi_3 \\ \xi_1^2 \left( \frac{L^{(i)}}{L_3} \right) \\ \dots \\ \vdots \end{array} \right)$$

Densities

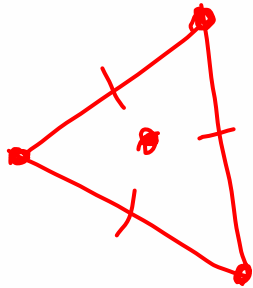
Uniendo los 3 vect de GPL:

$$\bar{r}^T = \left[ \omega_1 \theta_{x_1} \theta_{y_1} \quad \omega_2 \theta_{x_2} \theta_{y_2} \quad \omega_3 \theta_{x_3} \theta_{y_3} \quad \theta_4 \theta_5 \theta_6 \quad \vdots \quad \omega_0 \theta_{x_0} \theta_{y_0} \right]$$

$$= \left[ \bar{r}^T \quad \vdots \quad \bar{r}_0^T \right]$$

Uniendo P/ los 3 subelementos:

$$\begin{pmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \omega^{(3)} \end{pmatrix} = \begin{bmatrix} \phi_e^{(1)T} & \phi_0^{(1)T} \\ \phi_e^{(2)T} & \phi_0^{(2)T} \\ \phi_e^{(3)T} & \phi_0^{(3)T} \end{bmatrix} \begin{pmatrix} \bar{r} \\ \bar{r}_0 \end{pmatrix} \quad (*)$$

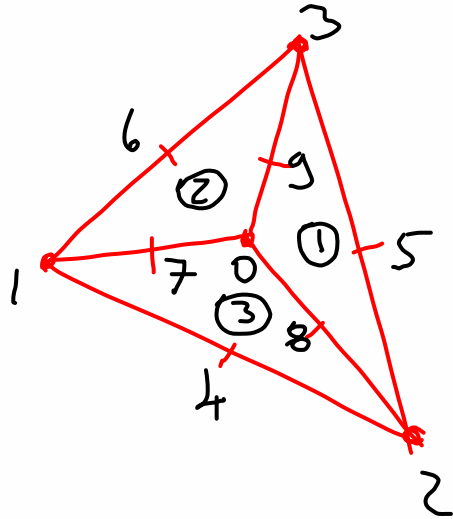


$$\omega^{(2)} = \phi_e^{(2)T} \bar{r} + \phi_0^{(2)T} \bar{r}_0$$

$$\frac{\partial \omega^{(2)}}{\partial n} \Big|_F = \frac{\partial \phi_e^{(2)T}}{\partial n} \bar{r} + \frac{\partial \phi_0^{(2)T}}{\partial n} \bar{r}_0$$



Regular compatibilidad interna  $C_1$ :



$$\left. \frac{\partial w^{(2)}}{\partial n} \right|_7 = \theta_7^{(2)} = -\theta_7^{(3)} = - \left. \frac{\partial w^{(3)}}{\partial n} \right|_7$$

Cambio de signo / conveni6n normal hacia afuera

$$\begin{pmatrix} \theta_7^{(2)} \\ \theta_8^{(3)} \\ \theta_9^{(1)} \end{pmatrix} + \begin{pmatrix} \theta_7^{(3)} \\ \theta_8^{(1)} \\ \theta_9^{(2)} \end{pmatrix} = \begin{bmatrix} \begin{matrix} |^{(2)T} & |^{(3)T} \\ \hline b_7 + b_7 & \vdots \\ \hline \end{matrix} & \begin{matrix} |^{(2)T} & |^{(3)T} \\ \hline b_{70} + b_{70} & \vdots \\ \hline \end{matrix} \\ \begin{matrix} |^{(3)T} & |^{(1)T} \\ \hline b_8 + b_8 & \vdots \\ \hline \end{matrix} & \begin{matrix} |^{(3)T} & |^{(1)T} \\ \hline b_{80} + b_{80} & \vdots \\ \hline \end{matrix} \\ \begin{matrix} |^{(1)T} & |^{(2)T} \\ \hline b_9 + b_9 & \vdots \\ \hline \end{matrix} & \begin{matrix} |^{(1)T} & |^{(2)T} \\ \hline b_{90} + b_{90} & \vdots \\ \hline \end{matrix} \end{bmatrix} \begin{pmatrix} r_7 \\ r_8 \\ r_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$b_7^{(2)} = \frac{\partial \phi_e}{\partial n} \dots$$

Metriaiolate:

$$\begin{bmatrix} \underline{\underline{B}} & \underline{\underline{B}}_0 \end{bmatrix} \begin{pmatrix} \underline{r} \\ \underline{r}_0 \end{pmatrix} = \underline{\underline{0}} \Rightarrow \underline{\underline{B}} \underline{r} + \underline{\underline{B}}_0 \underline{r}_0 = \underline{\underline{0}}$$

Deall:

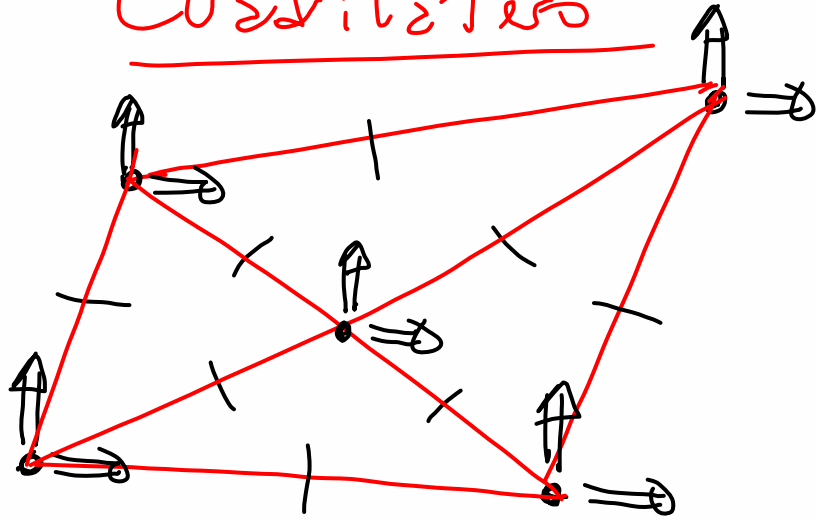
$$\underline{r}_0 = - \underline{\underline{B}}_0^{-1} \underline{\underline{B}} \underline{r} = \underline{\underline{L}} \underline{r}$$

Introducere (\*)

$$\begin{pmatrix} w^{(1)} \\ w^{(2)} \\ w^{(3)} \end{pmatrix} = \left( \begin{bmatrix} \phi_e^{(1)T} \\ \phi_e^{(2)T} \\ \phi_e^{(3)T} \end{bmatrix} + \begin{bmatrix} \phi_0^{(1)T} \\ \phi_0^{(2)T} \\ \phi_0^{(3)T} \end{bmatrix} \right) \underline{\underline{L}} \underline{r} = \begin{bmatrix} \hat{\phi}^{(1)T} \\ \hat{\phi}^{(2)T} \\ \hat{\phi}^{(3)T} \end{bmatrix} \underline{r}$$

$\left[ \hat{\phi}^{(i)T} \right]$

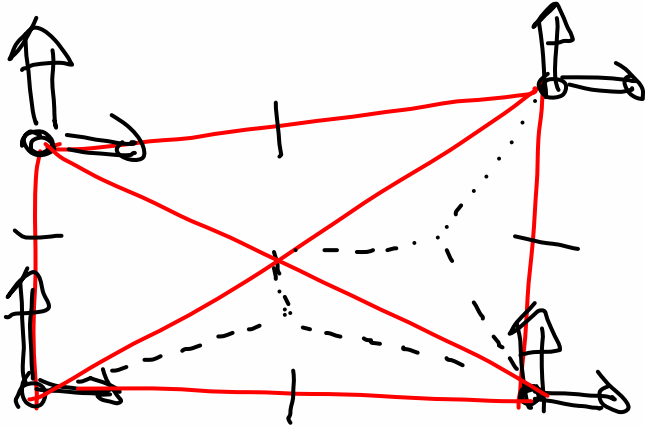
# Cuadrilátero



Unión de elementos LCCT-12

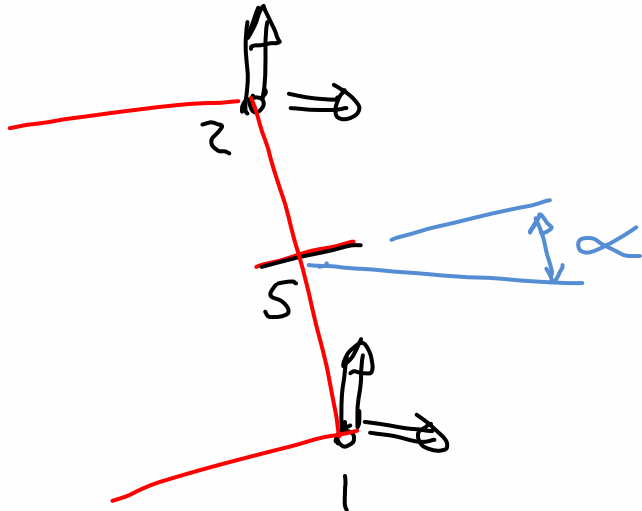
23 GDLs

Condensando los GDLs internos:



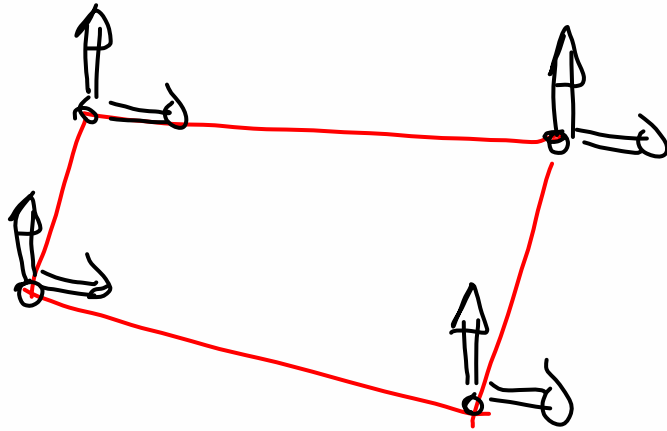
16 GDLs

Los GDLs de los 2 nudos de los lados se pueden eliminar haciendo un 2 hip cinemática adecuada:



$$\begin{aligned} \theta_5 &= \theta_{x_5} \cos \alpha + \theta_{y_5} \sin \alpha \approx \\ &= \frac{\theta_{x_1} + \theta_{x_2}}{2} \cos \alpha + \frac{\theta_{y_1} + \theta_{y_2}}{2} \sin \alpha \end{aligned}$$

Obtengo un elemento



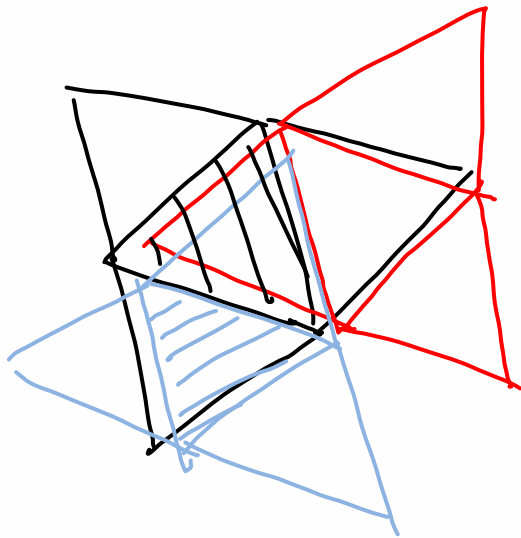
12 GDL

## Ejercicio:

- 1) Desarrollar el LCCT-12 usando un sig. subóptica y expresar los focos focal relativos.
- 2) Desarrollar un LCCT-g
- 3) " " " Q-12

Aplicar a la ecuación de Stokes.

∃ otros "patches" de electos:



(F. Flores)

Logica con  $\mathbb{C}_1$

## Desarrollo subelemento 10gd1

Aproximación de la geometría: polinomios lineales

$$\mathbf{x} = \mathbf{F}(\boldsymbol{\xi}) = \sum_{k=1}^3 \hat{\phi}_k(\boldsymbol{\xi}) \mathbf{a}^k, \quad \boldsymbol{\xi} \in \hat{K}$$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ 1 - \xi_1 - \xi_2 \end{bmatrix}$$

$$\mathbf{F}(\hat{\mathbf{x}}) = \begin{Bmatrix} F_1(\boldsymbol{\xi}) \\ F_2(\boldsymbol{\xi}) \end{Bmatrix} = \begin{bmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix}$$

$$\frac{\partial \mathbf{x}}{\partial \xi_j} = \frac{\partial \mathbf{F}}{\partial \xi_j}(\boldsymbol{\xi}) = \sum_{k=1}^3 \frac{\partial \hat{\phi}_k}{\partial \xi_j}(\boldsymbol{\xi}) \mathbf{x}^k$$

$$\mathbf{J}(\boldsymbol{\xi}) = \frac{\partial \mathbf{F}}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi}) = \begin{bmatrix} \frac{\partial F_1}{\partial \xi_1} & \frac{\partial F_1}{\partial \xi_2} \\ \frac{\partial F_2}{\partial \xi_1} & \frac{\partial F_2}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\phi}_1}{\partial \xi_1} & \frac{\partial \hat{\phi}_1}{\partial \xi_2} \\ \frac{\partial \hat{\phi}_2}{\partial \xi_1} & \frac{\partial \hat{\phi}_2}{\partial \xi_2} \\ \frac{\partial \hat{\phi}_3}{\partial \xi_1} & \frac{\partial \hat{\phi}_3}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} (x^1 - x^3) & (x^2 - x^3) \\ (y^1 - y^3) & (y^2 - y^3) \end{bmatrix}$$

## Desarrollo subelemento 10gdl (cont)

$$\mathbf{J}^{-1}(\xi) = \frac{\partial \xi}{\partial \mathbf{x}} = \frac{1}{2\Delta} \begin{bmatrix} (y^2 - y^3) & -(x^2 - x^3) \\ -(y^1 - y^3) & (x^1 - x^3) \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_1 & a_1 \\ b_2 & a_2 \end{bmatrix}$$

donde  $2\Delta = \det \mathbf{J}$   
(doble del área del triángulo)

Notar:  $2\Delta = \det \mathbf{J} = (x^1 - x^3)(y^2 - y^3) - (x^2 - x^3)(y^1 - y^3) = a_2 b_1 - a_1 b_2$



## Desarrollo subelemento 10gdl (cont)

Aproximación del campo: polinomio cúbico completo

$$w(\xi) = \sum_{0 \leq i+j \leq 3} c_{ij} \xi_1^i \xi_2^j = c_{00} + c_{10} \xi_1 + c_{01} \xi_2 + c_{20} \xi_1^2 + c_{11} \xi_1 \xi_2 + c_{02} \xi_2^2 + \dots = \mathbf{p}^T(\xi_1, \xi_2) \mathbf{c}, \quad \forall (\xi_1, \xi_2) \in \hat{K}$$

$$\mathbf{p}^T(\xi_1, \xi_2) = \left[ 1 \quad \xi_1 \quad \xi_2 \quad \xi_1^2 \quad \xi_1 \xi_2 \quad \xi_2^2 \quad \xi_1^3 \quad \dots \right]$$

$$\mathbf{c} = \left[ c_{00} \quad c_{10} \quad c_{01} \quad c_{20} \quad c_{11} \quad c_{02} \quad c_{30} \quad c_{21} \quad c_{12} \quad c_{03} \right]^T$$

Para el cálculo de los coeficientes  $\mathbf{c}$  expresamos el valor del campo en los

$$w(\mathbf{x}_k) = w(\xi_k) = \sum_{0 \leq i+j \leq 3} c_{ij} (\xi_1^k)^i (\xi_2^k)^j = c_{00} + c_{10} \xi_1^k + c_{01} \xi_2^k + c_{20} (\xi_1^k)^2 + c_{11} \xi_1^k \xi_2^k + c_{02} (\xi_2^k)^2 + \dots = w^k,$$

$$k = 1, 2, 3.$$

donde

$$\xi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \xi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \xi^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Obtenemos así tres ecuaciones con 10 incógnitas:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{c} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}$$

## Desarrollo subelemento 10gdl (cont)

Calculando luego el valor de la derivada respecto de  $x$  en los nodos, logramos tres ecuaciones más:

$$w_{,x}(\mathbf{x}^k) = \left( \frac{\partial w}{\partial \xi} \Big|_{\xi^k} \right) \frac{\partial \xi}{\partial x} \Big|_{x=x^k} = \left( \frac{\partial w}{\partial \xi} \Big|_{\xi^k} \right) \frac{1}{2\Delta} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{2\Delta} [b_1 \quad b_2] \frac{\partial w}{\partial \xi} \Big|_{\xi^k}$$

$$\frac{\partial w}{\partial \xi_1}(\xi) = \sum_{0 \leq i+j \leq 3} i c_{ij} \xi_1^{i-1} \xi_2^j = c_{10} + 2c_{20}\xi_1 + c_{11}\xi_2 + \dots = \frac{\partial \mathbf{p}^T}{\partial \xi_1} \mathbf{c}$$

$$\frac{\partial w}{\partial \xi_2}(\hat{\mathbf{x}}) = \sum_{0 \leq i+j \leq 3} j c_{ij} \xi_1^i \xi_2^{j-1} = c_{01} + c_{11}\xi_1 + 2c_{02}\xi_2 + \dots = \frac{\partial \mathbf{p}^T}{\partial \xi_2} \mathbf{c}$$

$$, \quad \forall (\xi_1, \xi_2) \in \hat{K}$$

$$\begin{bmatrix} \frac{\partial w}{\partial \xi_1} \\ \frac{\partial w}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2\xi_1 & \xi_2 & 0 & 3\xi_1^2 & 2\xi_1 \xi_2 & \xi_2^2 & 0 \\ 0 & 0 & 1 & 0 & \xi_1 & 2\xi_2 & 0 & \xi_1^2 & 2\xi_1 \xi_2 & 3\xi_2^2 \end{bmatrix} \mathbf{c}$$

Evaluando ambas derivadas (respecto de  $x$  y de  $y$ ) en los nodos:

$$\xi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \xi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \xi^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Obtenemos seis ecuaciones con 10 incógnitas:

## Desarrollo subelemento 10gdl (cont)

$$\frac{1}{2\Delta} \begin{bmatrix} b_1 & b_2 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & a_1 & a_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{c} = \begin{bmatrix} w_{,x}^1 \\ w_{,y}^1 \\ w_{,x}^2 \\ w_{,y}^2 \\ w_{,x}^3 \\ w_{,y}^3 \end{bmatrix}$$

$$\frac{1}{a_2b_1 - a_1b_2} \begin{bmatrix} 0 & b_1 & b_2 & 2b_1 & b_2 & 0 & 3b_1 & b_2 & 0 & 0 \\ 0 & a_1 & a_2 & 2a_1 & a_2 & 0 & 3a_1 & a_2 & 0 & 0 \\ 0 & b_1 & b_2 & 0 & b_1 & 2b_2 & 0 & 0 & b_1 & 3b_2 \\ 0 & a_1 & a_2 & 0 & a_1 & 2a_2 & 0 & 0 & a_1 & 3a_2 \\ 0 & b_1 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{c} = \begin{bmatrix} w_{,x}^1 \\ w_{,y}^1 \\ w_{,x}^2 \\ w_{,y}^2 \\ w_{,x}^3 \\ w_{,y}^3 \end{bmatrix}$$

## Desarrollo subelemento 10gdl (cont)

Finalmente, evaluando la derivada respecto de la normal en el punto intermedio entre los nodos 1 y 2:

$$w_{,n}(\mathbf{x}^{12}) = \frac{1}{2\Delta} (n_x [b_1 \quad b_2] + n_y [a_1 \quad a_2]) \left. \frac{\partial w}{\partial \xi} \right|_{\xi^{12}} = \frac{1}{2\Delta} [n_x b_1 + n_y a_1 \quad n_x b_2 + n_y a_2] \left. \frac{\partial w}{\partial \xi} \right|_{\xi^{12}} = w_{,n}^{12}$$

$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \frac{1}{\ell_{12}} \begin{bmatrix} y_2 - y_1 \\ x_1 - x_2 \end{bmatrix} \quad \ell_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\begin{bmatrix} \frac{\partial w}{\partial \xi_1} \\ \frac{\partial w}{\partial \xi_2} \end{bmatrix}_{\left(\frac{1}{2}, \frac{1}{2}\right)} = \begin{bmatrix} 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \mathbf{c}$$

Obtenemos así una décima ecuación:

$$\frac{1}{a_2 b_1 - a_1 b_2} \begin{bmatrix} 0 & d_1 & d_2 & d_1 & \frac{d_1 + d_2}{2} & d_2 & \frac{3}{4} d_1 & \frac{d_1 + d_2}{2} + \frac{d_2}{4} & \frac{d_1 + d_2}{4} + \frac{d_2}{2} & \frac{3}{4} d_2 \end{bmatrix} \mathbf{c} = w_{,n}^{12}$$

donde:

$$d_1 = n_x b_1 + n_y a_1; \quad d_2 = n_x b_2 + n_y a_2.$$

## Desarrollo subelemento 10gdl (cont)

Reordenando ecuaciones, tenemos el sistema:

$$\mathbf{A} \mathbf{c} = \frac{1}{2\Delta} \begin{bmatrix} 2\Delta & 2\Delta & 0 & 2\Delta & 0 & 0 & 2\Delta & 0 & 0 & 0 \\ 0 & b_1 & b_2 & 2b_1 & b_2 & 0 & 3b_1 & b_2 & 0 & 0 \\ 0 & a_1 & a_2 & 2a_1 & a_2 & 0 & 3a_1 & a_2 & 0 & 0 \\ 2\Delta & 0 & 2\Delta & 0 & 0 & 2\Delta & 0 & 0 & 0 & 2\Delta \\ 0 & b_1 & b_2 & 0 & b_1 & 2b_2 & 0 & 0 & b_1 & 3b_2 \\ 0 & a_1 & a_2 & 0 & a_1 & 2a_2 & 0 & 0 & a_1 & 3a_2 \\ 2\Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & d_2 & d_1 & \frac{d_1+d_2}{2} & d_2 & \frac{3}{4}d_1 & \frac{d_1+d_2}{2} & \frac{d_1+d_2}{4} & \frac{3}{4}d_2 \end{bmatrix} \mathbf{c} = \begin{bmatrix} w^1 \\ w_{,x}^1 \\ w_{,y}^1 \\ w^2 \\ w_{,x}^2 \\ w_{,y}^2 \\ w^3 \\ w_{,x}^3 \\ w_{,y}^3 \\ w_{,n}^{12} \end{bmatrix}$$

La matriz  $\mathbf{A}$  puede invertirse analíticamente y calcular así las funciones de forma:

$$\varphi(\xi_1, \xi_2) = \mathbf{p}^T(\xi_1, \xi_2) \mathbf{A}^{-1}$$

## Desarrollo subelemento 10gdl (cont)

Se obtiene:

$$\left\{ \begin{array}{l} \varphi_1(\xi_1, \xi_2, \xi_3) = \xi_1^2(3 - 2\xi_1) + 6\mu_3\xi_1\xi_2\xi_3 \\ \varphi_2(\xi_1, \xi_2, \xi_3) = -\xi_1^2(a_3\xi_2 - a_2\xi_3) - (a_3\mu_3 - a_1)\xi_1\xi_2\xi_3 \\ \varphi_3(\xi_1, \xi_2, \xi_3) = -\xi_1^2(b_3\xi_2 - b_2\xi_3) + (b_3\mu_3 - b_1)\xi_1\xi_2\xi_3 \\ \varphi_4(\xi_1, \xi_2, \xi_3) = \xi_2^2(3 - 2\xi_2) + 6\lambda_3\xi_1\xi_2\xi_3 \\ \varphi_5(\xi_1, \xi_2, \xi_3) = -\xi_2^2(a_1\xi_3 - a_3\xi_1) + (a_3\lambda_3 - a_2)\xi_1\xi_2\xi_3 \\ \varphi_6(\xi_1, \xi_2, \xi_3) = \xi_2^2(b_1\xi_3 - b_3\xi_1) - (b_3\lambda_3 - b_2)\xi_1\xi_2\xi_3 \\ \varphi_7(\xi_1, \xi_2, \xi_3) = \xi_3^2(3 - 2\xi_3) \\ \varphi_8(\xi_1, \xi_2, \xi_3) = -\xi_3^2(a_2\xi_1 - a_1\xi_2) \\ \varphi_9(\xi_1, \xi_2, \xi_3) = \xi_3^2(b_2\xi_1 - b_1\xi_2) \\ \varphi_{10}(\xi_1, \xi_2, \xi_3) = 4H_3\xi_1\xi_2\xi_3 \end{array} \right.$$

donde:

$$\mu_3 = \frac{d_1}{d_1 + d_2};$$

$$\lambda_3 = \frac{d_2}{d_1 + d_2};$$

$$H_3 = \frac{-1}{d_1 + d_2}$$

y:

$$\xi_3 = 1 - \xi_1 - \xi_2$$

Se adjunta a continuación código de manipulación simbólica para cálculo

```

clear
syms x1 x2 x3 y1 y2 y3 real
syms xi1 xi2 positive
phih1 = xi1;
phih2 = xi2;
phih3 = 1 - xi1 - xi2;

AA = [x1 x2 x3
      y1 y2 y3];

Dphih = [eval(diff(phih1,xi1)) eval(diff(phih1,xi2));
         eval(diff(phih2,xi1)) eval(diff(phih2,xi2));
         eval(diff(phih3,xi1)) eval(diff(phih3,xi2)); ];

J = AA * Dphih;

J1 = inv(J);

% b1/(a1*b2-a2*b1) <- J1(1,1);
% a1/(a1*b2-a2*b1) <- J1(1,2);
% b2/(a1*b2-a2*b1) <- J1(2,1);
% a2/(a1*b2-a2*b1) <- J1(2,2);

syms b1 a1 b2 a2 n1 n2 real

p = 1;
for i=1:3
    for j=0:i
        p = [p xi1^(i-j)*xi2^j];
    end
end
end
%

```

```

dpxi1 = subs(pxi1,{xi1,xi2},{0,1});
dpxi2 = subs(pxi2,{xi1,xi2},{0,1});
C(5,:) = b1/(a1*b2-a2*b1)*dpxi1 + b2/(a1*b2-a2*b1)*dpxi2;

dpxi1 = subs(pxi1,{xi1,xi2},{0,0});
dpxi2 = subs(pxi2,{xi1,xi2},{0,0});
C(8,:) = b1/(a1*b2-a2*b1)*dpxi1 + b2/(a1*b2-a2*b1)*dpxi2;

%
% dw/dx2 (ak1,ak2) = [a1 a2]* dw/dxh (xi1^k,xi2^k) = dwx1_k k=1,3
%
pxi1 = diff(p,xi1);
pxi2 = diff(p,xi2);
dpxi1 = subs(pxi1,{xi1,xi2},{1,0});
dpxi2 = subs(pxi2,{xi1,xi2},{1,0});
C(3,:) = a1/(a1*b2-a2*b1)*dpxi1 + a2/(a1*b2-a2*b1)*dpxi2;

dpxi1 = subs(pxi1,{xi1,xi2},{0,1});
dpxi2 = subs(pxi2,{xi1,xi2},{0,1});
C(6,:) = a1/(a1*b2-a2*b1)*dpxi1 + a2/(a1*b2-a2*b1)*dpxi2;

dpxi1 = subs(pxi1,{xi1,xi2},{0,0});
dpxi2 = subs(pxi2,{xi1,xi2},{0,0});
C(9,:) = a1/(a1*b2-a2*b1)*dpxi1 + a2/(a1*b2-a2*b1)*dpxi2;

%
% dw/dn = [n1 n2]*[ dw/dx1(a^{23}) dw/dx2(a^{23})] =
% l23 = sqrt((y3-y2)^2 +(x3-x2)^2);
% n = [(y3-y2); -(x3-x2)]/l23;

dpxi1 = subs(pxi1,{xi1,xi2},{1/2,1/2});
dpxi2 = subs(pxi2,{xi1,xi2},{1/2,1/2});

```



```
C(10,:)= [0, c1, c2, c1, (c1+c2)/2, c2, 3*c1/4, c1/2 + c2/4, c1/4 + c2/2, 3*c2/4];
```

```
Cinv = inv(C);
```

```
phi1 = p*Cinv(:,1);
```

```
phi2 = p*Cinv(:,2);
```

```
phi3 = p*Cinv(:,3);
```

```
phi4 = p*Cinv(:,4);
```

```
phi5 = p*Cinv(:,5);
```

```
phi6 = p*Cinv(:,6);
```

```
phi7 = p*Cinv(:,7);
```

```
phi8 = p*Cinv(:,8);
```

```
phi9 = p*Cinv(:,9);
```

```
phi10 = p*Cinv(:,10);
```

```
syms mu3 lam3 xi3 b3 a3 H3
```

```
phi1Fel = xi1^2*(3-2*xi1) + 6*mu3*xi1*xi2*xi3;
```

```
phi1Fel = subs(phi1Fel,mu3,c1/(c1+c2));
```

```
phi1Fel = subs(phi1Fel,xi3,1-xi1-xi2);
```

```
errphi1 = simplify(phi1-phi1Fel)
```

```
phi2Fel = - xi1^2*(a3*xi2-a2*xi3) - (a3*mu3-a1)*xi1*xi2*xi3;
```

```
phi2Fel = subs(phi2Fel,a3,-a1-a2);
```

```
phi2Fel = subs(phi2Fel,mu3,c1/(c1+c2));
```

```
phi2Fel = subs(phi2Fel,xi3,1-xi1-xi2);
```

```
errphi2 = simplify(phi2-phi2Fel)
```

```
phi3Fel = xi1^2*(b3*xi2-b2*xi3) + (b3*mu3-b1)*xi1*xi2*xi3;
```

```
phi5Fel = - xi2^2*(a1*xi3-a3*xi1) + (a3*lam3-a2)*xi1*xi2*xi3;
```

```
phi5Fel = subs(phi5Fel,a3,-a1-a2);
```

```
phi5Fel = subs(phi5Fel,lam3,c2/(c1+c2));
```

```
phi5Fel = subs(phi5Fel,xi3,1-xi1-xi2);
```

```
errphi5 = simplify(phi5-phi5Fel)
```

```
phi6Fel = xi2^2*(b1*xi3-b3*xi1) - (b3*lam3-b2)*xi1*xi2*xi3;
```

```
phi6Fel = subs(phi6Fel,b3,-b1-b2);
```

```
phi6Fel = subs(phi6Fel,lam3,c2/(c1+c2));
```

```
phi6Fel = subs(phi6Fel,xi3,1-xi1-xi2);
```

```
errphi6 = simplify(phi6-phi6Fel)
```

```
phi7Fel = xi3^2*(3-2*xi3);
```

```
phi7Fel = subs(phi7Fel,xi3,1-xi1-xi2);
```

```
errphi7 = simplify(phi7-phi7Fel)
```

```
phi8Fel = - xi3^2*(a2*xi1-a1*xi2);
```

```
phi8Fel = subs(phi8Fel,xi3,1-xi1-xi2);
```

```
errphi8 = simplify(phi8-phi8Fel)
```

```
phi9Fel = xi3^2*(b2*xi1-b1*xi2);
```

```
phi9Fel = subs(phi9Fel,xi3,1-xi1-xi2);
```

```
errphi9 = simplify(phi9-phi9Fel)
```

```
phi10Fel = 4*H3*xi1*xi2*xi3;
```

```
phi10Fel = subs(phi10Fel,xi3,1-xi1-xi2);
```

