Approximation to PDE
by continuous trial functions

- Defining the PDE problem
- Introduction to the PDE approximation
- Approximation by weighted residual
- Simultaneous approximation to the solution of the PDE in the domain and on its boundary.
- Natural boundary conditions
- Boundary solution methods
- Systems of PDE
- Nonlinear problems

Defining the PDE problem

$$A(f) = 0 \qquad \text{in } \Omega$$

L : linear differential operator

$$p \neq p(f)$$

$$\mathsf{L}(f) = \frac{\partial}{\partial x}\left(k\,\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial f}{\partial y}\right)$$

$$p = Q$$

$$k \neq k(f)$$

$$B(f) = \mathsf{M}(f) + \mathrm{r} = 0 \qquad \text{in } \Gamma$$

$$\begin{cases} \mathsf{M}(f) = f & ; \quad r = -\overline{f} \qquad \text{on } \Gamma_f \quad \text{DIRICHLET} \\ \mathsf{M}(f) = -k\,\dfrac{\partial f}{\partial h} & ; \quad r = -\overline{q} \qquad \text{on } \Gamma_q \quad \text{NEUMANN} \end{cases}$$

Introduction to the PDE approximation
How to choose the trial functions

$$f(x) \cong \hat{f}(x) = y(x) + \sum_m a_m N_m(x)$$

$$\text{on } \Gamma \quad \begin{cases} \mathrm{M}(y) = -r \\ \mathrm{M}(N_m) = 0 \quad ; \quad m = 1, 2, \cdots \end{cases}$$

$$\hat{f}(x) = f(x) \quad \text{for all } x \in \Gamma$$

Introduction to the PDE approximation
The trial functions and its derivatives

$$f(x) \cong \hat{f}(x) = y(x) + \sum_m a_m N_m(x)$$

$$\mathsf{L}\hat{f} = \mathsf{L}y + \sum_m a_m \mathsf{L}N_m$$

$$\frac{\partial f}{\partial x} \cong \frac{\partial \hat{f}}{\partial x} = \frac{\partial y}{\partial x} + \sum_m a_m \frac{\partial N_m}{\partial x}$$

$$\frac{\partial^2 f}{\partial x} \cong \frac{\partial^2 \hat{f}}{\partial x} = \frac{\partial^2 y}{\partial x} + \sum_m a_m \frac{\partial^2 N_m}{\partial x}$$

N_m has the regularity imposed by L

Approximation by weighted residual

$$R_{\Omega} = A(f) = \mathsf{L}\hat{f} + p = \mathsf{L}y + \sum_{m} a_m \mathsf{L} N_m + p$$

$$\int_{\Omega} W_l \; R_{\Omega} \; d\Omega = \int_{\Omega} W_l \left(\mathsf{L}y + \sum_{m} a_m \mathsf{L} N_m + p \right) d\Omega = 0$$

$$\int_{\Omega} W_l \left(\sum_{m} a_m \mathsf{L} N_m \right) d\Omega = -\int_{\Omega} W_l \left(\mathsf{L}y + p \right) d\Omega$$

$$K_{lm} a_m = f_l$$

$$K_{lm} = \int_{\Omega} W_l \mathsf{L} N_m \; d\Omega$$

$$f_l = -\int_{\Omega} W_l \left(\mathsf{L}y + p \right) d\Omega \qquad l, m = 1, 2, \cdots, M$$

Point collocation

Weight function $W_l = d(x - x_l)$

$$\int_\Omega W_l R_\Omega d\Omega = \int_\Omega d(x - x_l)\left(L\hat{f} + p\right) d\Omega =$$

$$= \int_\Omega d(x - x_l)\left(\left(Ly + \sum_m a_m L N_m\right) + p\right) d\Omega$$

$$\therefore \quad \int_\Omega d(x - x_l)\left(\sum_m a_m L N_m\right) d\Omega = -\int_\Omega d(x - x_l)\left(Ly + p\right) d\Omega$$

$$K_{lm}\; a_m \;=\; f_l$$

$$K_{lm} = \int_\Omega d(x - x_l)\, L N_m \; d\Omega = L N_m (x = x_l)$$

$$f_l = -\int_\Omega d(x - x_l)\left(Ly + p\right) d\Omega = -\left(Ly\,(x = x_l) + p(x = x_l)\right)$$

Subdomain collocation

Weight function $$W_l = c_{\Omega_l} = \begin{cases} 1 & x \in \Omega_l \\ 0 & x \notin \Omega_l \end{cases}$$

$$\int_\Omega W_l R_\Omega d\Omega = \int_\Omega c_{\Omega_l} \left(L\hat{f} + p \right) d\Omega =$$

$$= \int_\Omega c_{\Omega_l} \left(\left(L y + \sum_m a_m L N_m \right) + p \right) d\Omega$$

$$\therefore \quad \int_\Omega c_{\Omega_l} \left(\sum_m a_m L N_m \right) d\Omega = -\int_\Omega c_{\Omega_l} \left(L y + p \right) d\Omega$$

$$K_{lm}\, a_m = f_l$$

$$K_{lm} = \int_\Omega c_{\Omega_l}\, L N_m \, d\Omega = \int_{\Omega_l} L N_m d\Omega$$

$$f_l = -\int_\Omega c_{\Omega_l} \left(L y + p \right) d\Omega = -\int_{\Omega_l} \left(L y + p \right) d\Omega$$

Galerkin

Weight function $W_l = N_l$

$$\int_\Omega W_l R_\Omega d\Omega = \int_\Omega N_l \left(\mathsf{L}\hat{f} + p \right) d\Omega =$$

$$= \int_\Omega N_l \left(\left(\mathsf{L}y + \sum_m a_m \mathsf{L} N_m \right) + p \right) d\Omega$$

$$\therefore \quad \int_\Omega N_l \left(\sum_m a_m \mathsf{L} N_m \right) d\Omega = -\int_\Omega N_l \left(\mathsf{L}y + p \right) d\Omega$$

$$K_{lm} \, a_m = f_l$$

$$K_{lm} = \int_\Omega N_l \mathsf{L} N_m \, d\Omega$$

$$f_l = -\int_\Omega N_l \left(\mathsf{L}y + p \right) d\Omega$$

Example 1- Definition

Find $\hat{f}(x)$ solution of the following ODE

$$\frac{d^2 f}{dx^2} - f = 0 \qquad \text{in } \Omega : \{x; 0 \le x \le 1\}$$

$$f(x=0) = 0$$

$$f(x=1) = 1 \qquad \text{in } \Gamma : \{x = 0, x = 1\}$$

$$\mathsf{L}(f) = \frac{d^2 f}{dx^2} - f$$

$$p = 0$$

$$\mathsf{M}(f) + r = 0 \qquad \text{in } \Gamma$$

$$\begin{cases} \mathsf{M}(f) = f & ; \quad r = 0 \qquad \text{on } x = 0 \quad \text{DIRICHLET} \\ \mathsf{M}(f) = f & ; \quad r = -1 \qquad \text{on } x = 1 \quad \text{DIRICHLET} \end{cases}$$

Example 1- Choosing the approximation

$$f(x) \cong \hat{f}(x) = y\,(x) + \sum_m a_m N_m(x)$$

$$y\,(x) = x$$

$$N_m(x) = \sin(mp\;\, x)$$

$$y\,(x=0) = f(x=0) = 0$$

$$N_m(x=0) = 0$$

$$y\,(x=1) = f(x=1)\;\, = 1$$

$$N_m(x=1) = 0$$

$$\mathsf{L}\hat{f} = \mathsf{L}y\;\, + \sum_m a_m \mathsf{L} N_m$$

$$\mathsf{L}y\;\, = \frac{d^2 y}{dx^2} - y\;\, = 0 - x = -x$$

$$\mathsf{L}N_m = \frac{d^2 N_m}{dx^2} - N_m = (mp\,)^2(-\sin(mp\;\, x)) - \sin(mp\;\, x)$$

Example 1- Weighted residual using collocation

$$K_{lm}\ a_m\ =\ f_l$$

$$K_{lm} = \mathrm{L} N_m (x = x_l)$$

$$\mathrm{L} N_m = \sin(mp\ x)\left(-1-(mp)^2\right)$$

$$K_{lm} = \sin(mp\ x_l)\left(-1-(mp)^2\right)$$

$$\mathrm{L}\, y = -x \qquad ; \qquad p = 0$$

$$f_l = -\left(-x_l + 0\right) = x_l$$

$$\therefore \qquad a_m = \left(K^{-1} f\right)_m$$

Example 1- Weighted residual using collocation (see routine Ej_2_1.m)

$$K_{lm}\, a_m \;=\; f_l$$

$$K_{lm} = -\sin(mp\; x_l)\left(1+(mp)^2\right) \qquad ; \qquad f_l = x_l$$

$$\therefore K = -\left|\begin{array}{cccc} \sin(p\; x_1)\left(1+p^2\right) & \sin(2p\; x_1)\left(1+4p^2\right) & \cdots & \sin(Mp\; x_1)\left(1+M^2p^2\right) \\ \sin(p\; x_2)\left(1+p^2\right) & \sin(2p\; x_2)\left(1+4p^2\right) & \cdots & \sin(Mp\; x_2)\left(1+M^2p^2\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(p\; x_M)\left(1+p^2\right) & \sin(2p\; x_M)\left(1+4p^2\right) & & \sin(Mp\; x_M)\left(1+M^2p^2\right) \end{array}\right|$$

$$f = \left|x_1 \quad x_2 \quad \cdots \quad x_M\right|^T$$

Example 1- Weighted residual using Galerkin (see routine Ej_2_1.m)

$$K_{lm}\ a_m\ =\ f_l$$

$$K_{lm} = -\int_0^1 \sin(lp\ x)\,\sin(mp\ x)\left(1+(mp)^2\right)dx \quad ; \quad f_l\ = \int_0^1 x\sin(lp\ x)dx$$

$$\therefore K = -\left|\begin{matrix} \int_0^1 \sin(p\ x)\,\sin(p\ x)\left(1+p^{2}\right)dx & \int_0^1 \sin(p\ x)\,\sin(2p\ x)\left(1+4p^{2}\right)dx & \cdots & \int_0^1 \sin(p\ x)\,\sin(Mp\ x)\left(1+M^{2}p^{2}\right)dx \\ \int_0^1 \sin(2p\ x)\,\sin(p\ x)\left(1+p^{2}\right)dx & \int_0^1 \sin(2p\ x)\,\sin(2p\ x)\left(1+4p^{2}\right)dx & \cdots & \int_0^1 \sin(2p\ x)\,\sin(Mp\ x)\left(1+M^{2}p^{2}\right)dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 \sin(Mp\ x)\,\sin(p\ x)\left(1+p^{2}\right)dx & \int_0^1 \sin(Mp\ x)\,\sin(2p\ x)\left(1+4p^{2}\right)dx & & \int_0^1 \sin(Mp\ x)\,\sin(Mp\ x)\left(1+M^{2}p^{2}\right)dx \end{matrix}\right|$$

$$f\ = \left|\begin{matrix} \int_0^1 x\sin(p\ x)dx & \int_0^1 x\sin(2p\ x)dx & \cdots & \int_0^1 x\sin(Mp\ x)dx \end{matrix}\right|^{T}$$

since $\displaystyle\int_0^1 \sin(lp\ x)\,\sin(mp\ x)dx\ =\ \frac{1}{2}d_{lm}$ by orthogonality

and $\displaystyle\int_0^1 x\sin(lp\ x)\,dx\ =\ \left(-1\right)^{l+1}\frac{1}{lp}$

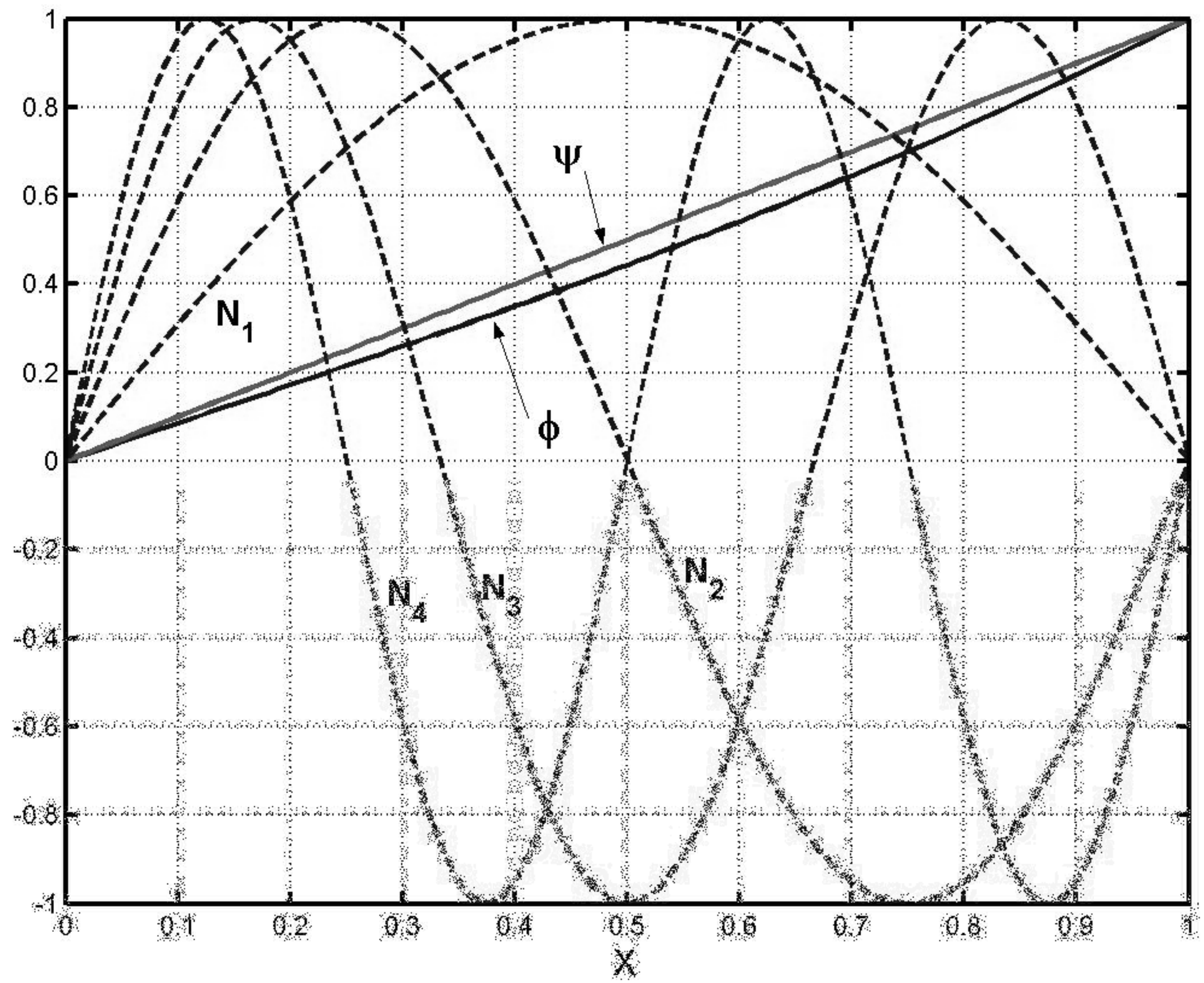
$$\therefore\quad \frac{1}{2}d_{lm}\left(1+m^2p^{2}\right)a_m = \left(-1\right)^m\frac{1}{mp} \quad\Rightarrow\quad a_m = \left(-1\right)^m\frac{2}{mp\left(1+m^2p^{2}\right)}$$

Example 1- Exact solution and approximation basis

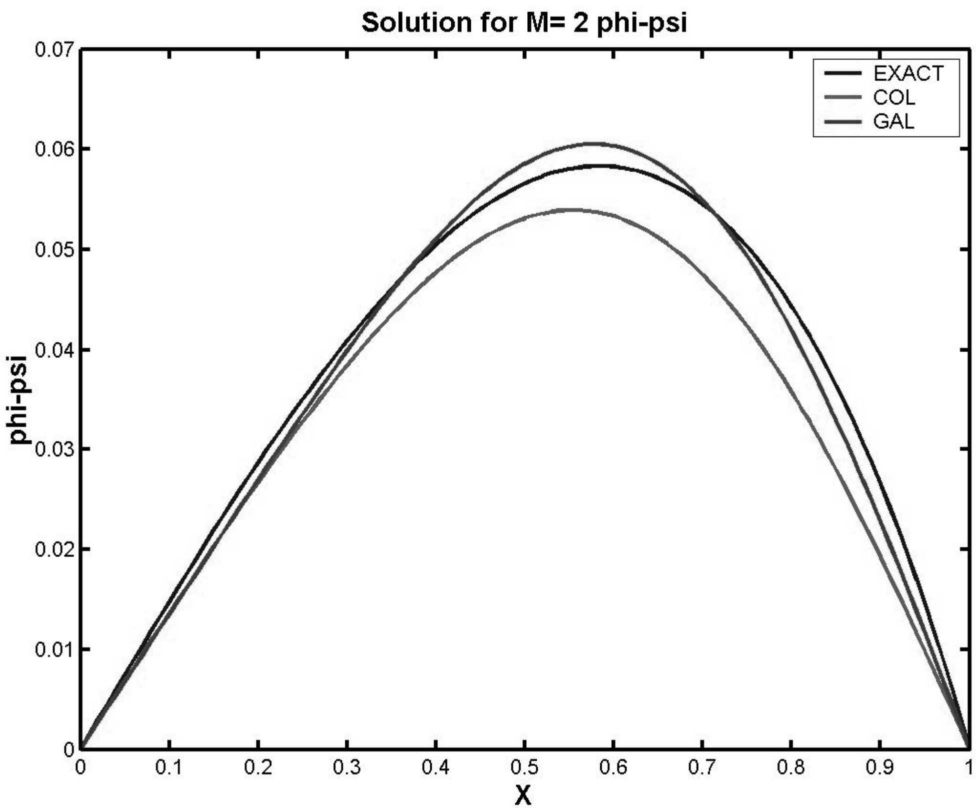
$$f^{\text{exact}} = \frac{e^{x} - e^{-x}}{e^{1} - e^{-1}}$$

$$N_m = \sin(mp\ x)$$

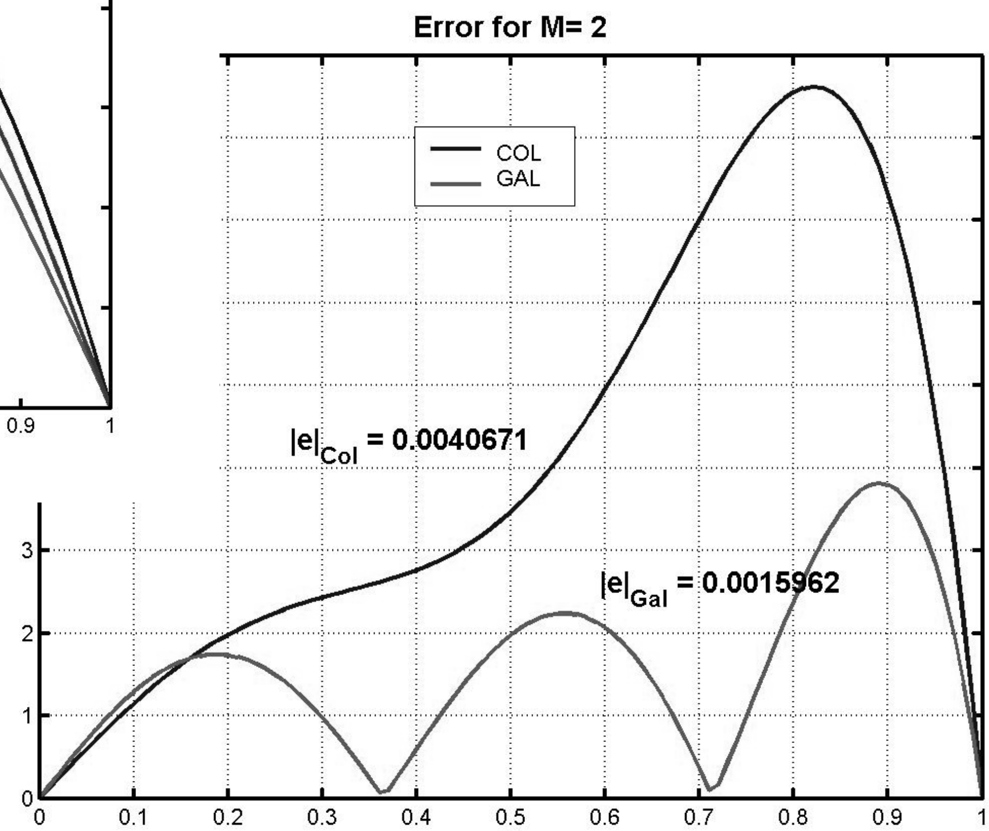
$$y = x$$



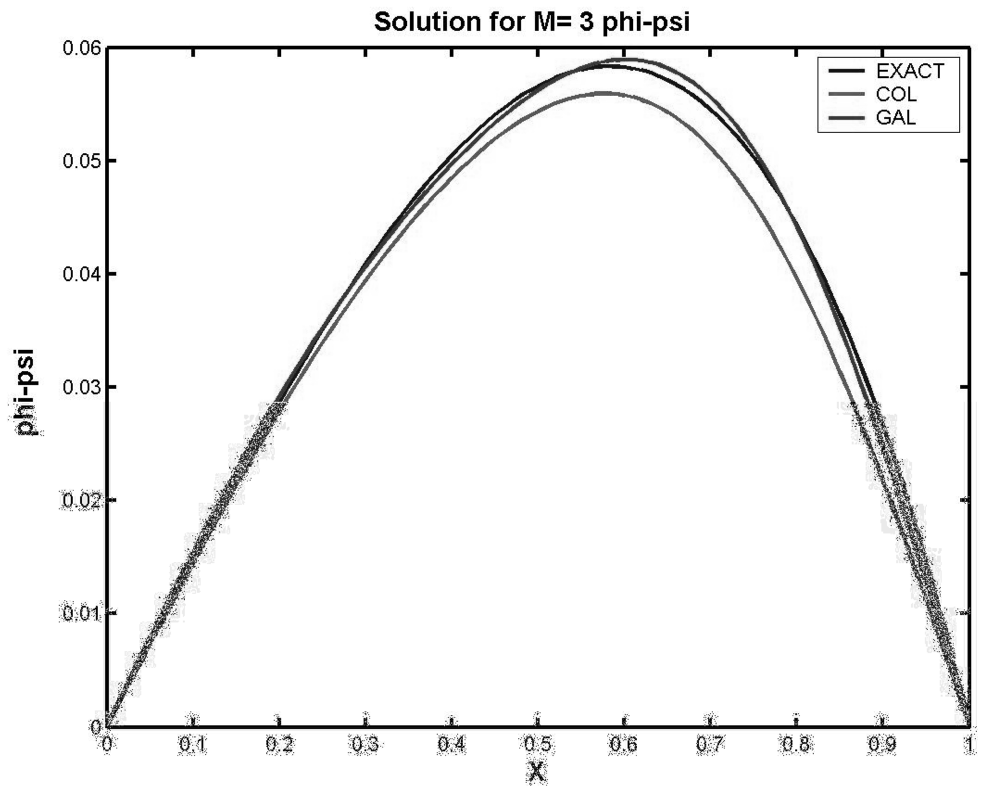
Example 1- Results using point collocation & Galerkin



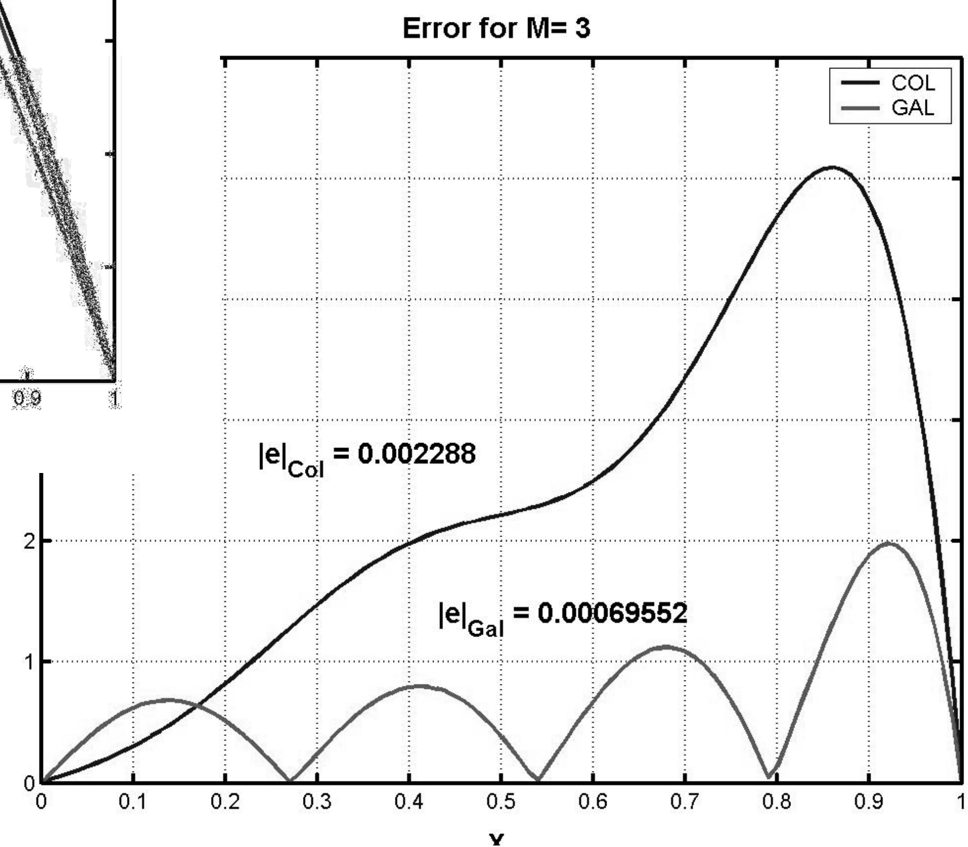
using M = 2



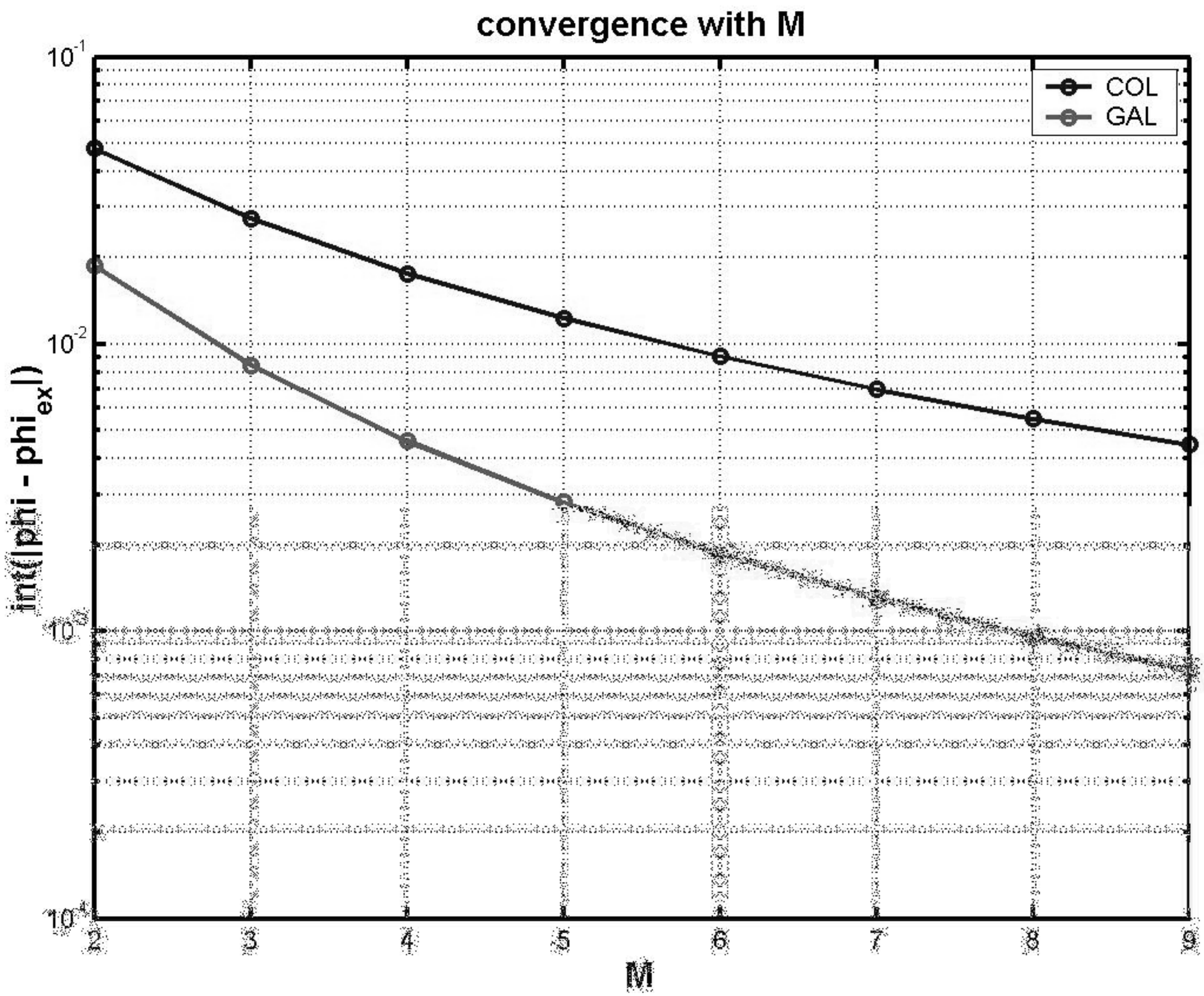
Example 1- Results using point collocation & Galerkin



using M = 3



Example 1- Results using point collocation & Galerkin



Example 2- A 2D scalar elastic problem - Definition

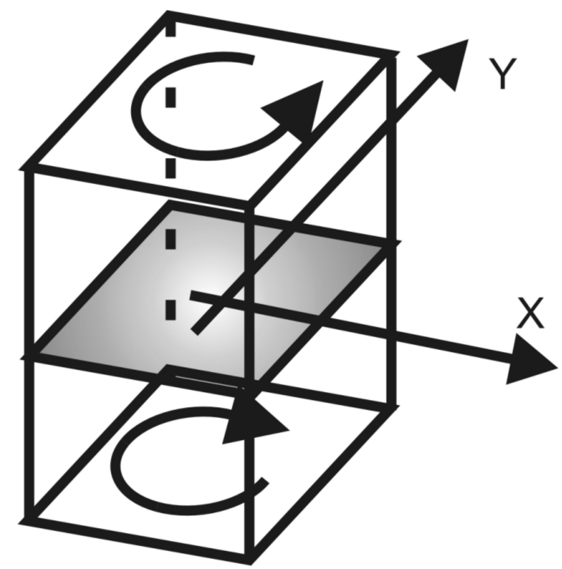
Find $\hat{f}(x)$ solution of the following PDE

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=-2Gq \qquad \text{in } \Omega:\{(x,y);\ -3\le x\le 3;\ -2\le y\le 2\}$$

$$f(x=-3)=f(x=3)=0$$

$$f(y=-2)=f(y=2)=0$$

$Gq=1$ G : shear modulus ; q : torsion angle



$$\mathrm{L}(f)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

$$p=2$$

$$\mathrm{M}(f)=f \quad ; \quad r=0$$

on $(x=-3)\cup(x=3)\cup(y=-2)\cup(y=2)$ DIRICHLET

Example 2- Choosing the approximation

$$f(x,y) \cong \hat{f}(x,y) = y\,(x,y) + \sum_m a_m N_m(x,y)$$

$$y\,(x,y) = 0 \quad \text{satisfying the boundary condition}$$

$$N_m(x,y) = \cos(a_x(m)\frac{p\,x}{6})\cos(a_y(m)\frac{p\,y}{4})$$

$$a = \left(a_x, a_y\right) = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$N_m(x,y) = 0 \quad \forall \left(\mathrm{x},\mathrm{y}\right) \in \Gamma$$

$$R_\Omega = \mathsf{L}\hat{f} + p = \mathsf{L}y + \sum_m a_m \mathsf{L} N_m + p$$

$$= \sum_m a_m \mathsf{L} N_m + p$$

$$\mathsf{L} N_m = \frac{\partial^2 N_m}{\partial x^2} + \frac{\partial^2 N_m}{\partial y^2} = -p^2\left(\left(\frac{a_x}{6}\right)^2 + \left(\frac{a_y}{4}\right)^2\right) N_m$$

Example 2- Weighted residual using Galerkin (see routine Ej_2_2.m)

$$\int_\Omega W_l R_\Omega d\Omega = \int_\Omega W_l \left(\mathsf{L}\hat{f} + p \right) d\Omega = 0$$

$$\int_\Omega N_l \left(\sum_m a_m \mathsf{L} N_m + p \right) d\Omega = 0$$

$$\int_\Omega N_l \left(\sum_m a_m \left(-p^2 \left(\left(\frac{a_x(m)}{6} \right)^2 + \left(\frac{a_y(m)}{4} \right)^2 \right) N_m \right) + p \right) d\Omega = 0$$

$$K_{lm}\ a_m\ =\ f_l$$

$$K_{lm} = -p^2 \left(\left(\frac{a_x(m)}{6} \right)^2 + \left(\frac{a_y(m)}{4} \right)^2 \right) \int_\Omega N_l N_m d\Omega$$

$$= -p^2 \left(\left(\frac{a_x(m)}{6} \right)^2 + \left(\frac{a_y(m)}{4} \right)^2 \right) \int_{-2}^{2} dy \int_{-3}^{3} N_l N_m dx$$

$$f_l\ = -\int_\Omega N_l p\ d\Omega = -2\int_\Omega N_l\ d\Omega = -2\int_{-2}^{2} dy \int_{-3}^{3} N_l\ dx$$

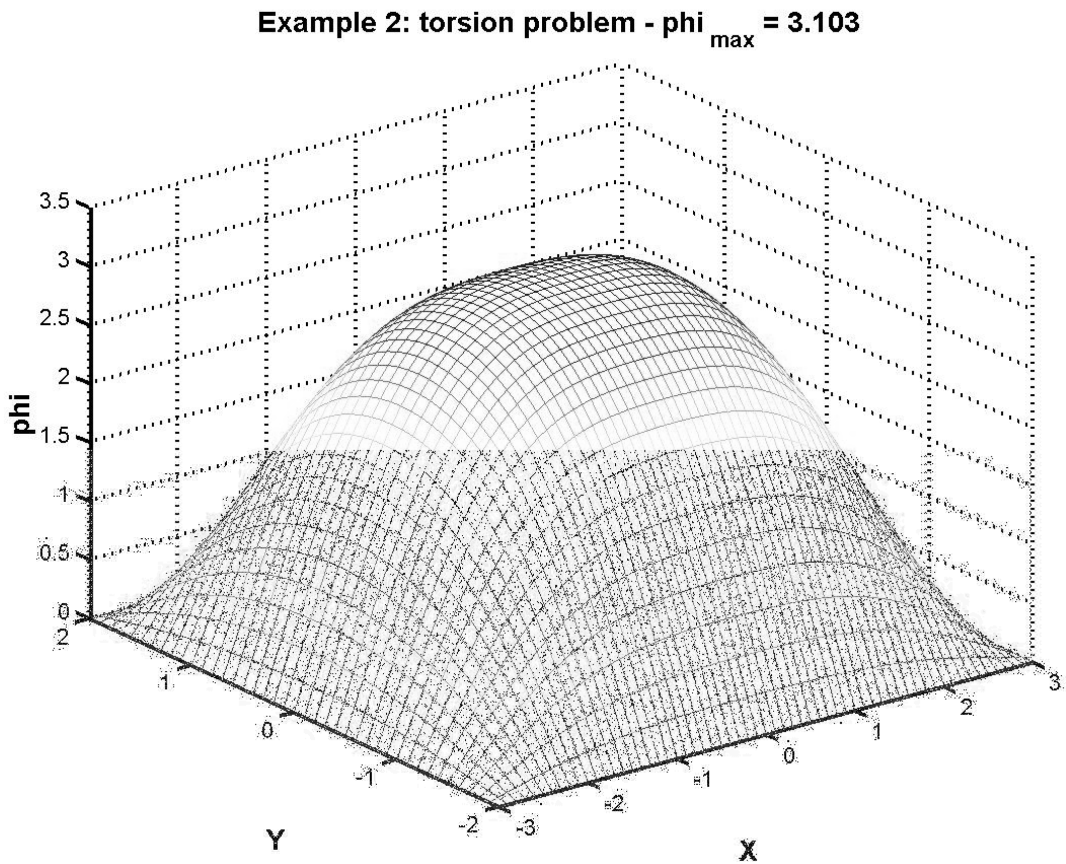
Example 2- Weighted residual using Galerkin (see routine Ej_2_2_symb.m with symbolic soft)

```
syms ax ay x y

ax=[1,3,1]
ay=[1,1,3]

N = [ cos(ax(1)*pi*x/6)*cos(ay(1)*pi*y/4);
      cos(ax(2)*pi*x/6)*cos(ay(2)*pi*y/4);
      cos(ax(3)*pi*x/6)*cos(ay(3)*pi*y/4)];

for l=1:3,
   for m=1:3,
      factor =    pi^2*((ax(m)/6)^2+(ay(m)/4)^2);
      K(l,m) = factor*int(int(N(l)*N(m),x,-3,3),y,-2,2);
   end
   f(l) = -2*int(int(N(l),x,-3,3),y,-2,2);
end
```

Example 2- Weighted residual using Galerkin (see routine Ej_2_2_num.m)

```
global X Y ll mm coef

% Number of terms to be used (M)
M = 3;
coef.p = 2;
% grid of samples
[X,Y]=meshgrid(-3:0.1:3,-2:0.1:2);

f = zeros(M,1);
K = zeros(M,M);
for ll=1:M
    f(ll) = gauss_integration('ffun_Ej_2_2_num_rhs',-3,3,-2,2);
    for mm=1:M
        K(ll,mm) = gauss_integration('ffun_Ej_2_2_num_lhs',-3,3,-2,2);
    end
end

a = K\f;
```

Example 2- Weighted residual using Galerkin Results



$$a_1 = \frac{4608}{13\,p^4} \quad ; \quad a_2 = -\frac{4608}{135\,p^4} \quad ; \quad a_3 = -\frac{4608}{255\,p^4}$$

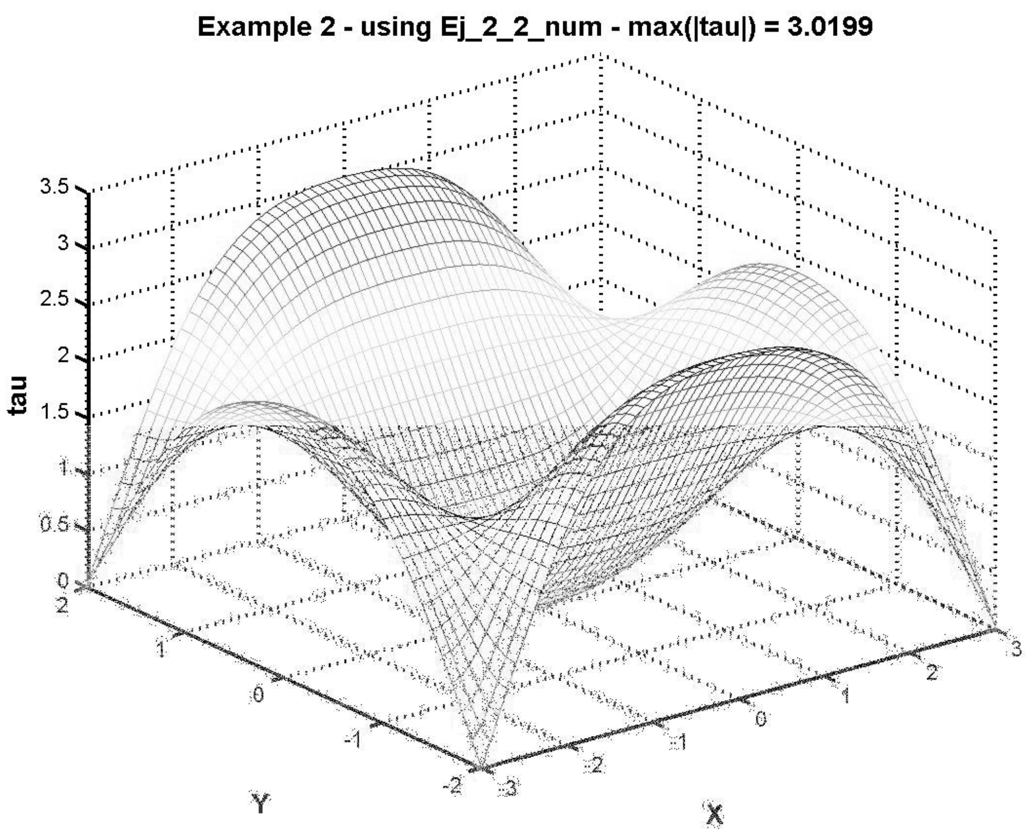
$$T = 2\int_{-2}^{2} dy \int_{-3}^{3} \hat{f}\, dx = 74.265 \qquad \text{(exact value 76.4)}$$

Example 2- Weighted residual using Galerkin Results

$$t_k = \frac{\partial f}{\partial x_k} \quad ; \quad |t| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

$$|\hat{t}| = \sqrt{\left(\frac{\partial \hat{f}}{\partial x}\right)^2 + \left(\frac{\partial \hat{f}}{\partial y}\right)^2} = \sqrt{\left(\sum_m a_m \frac{\partial N_m}{\partial x}\right)^2 + \left(\sum_m a_m \frac{\partial N_m}{\partial y}\right)^2}$$

3.02 (exact value 2.96)



Least square approximation to PDE

$$I(a_1, a_2, a_3, \cdots, a_M) = \int_\Omega R_\Omega^{\ 2} d\Omega$$

$$\frac{\partial I}{\partial a_l} = 0 \qquad l = 1,2,\cdots,M$$

$$\frac{\partial \int_\Omega R_\Omega^{\ 2} d\Omega}{\partial a_l} = \int_\Omega \frac{\partial}{\partial a_l} R_\Omega^{\ 2} d\Omega = \int_\Omega 2 R_\Omega \frac{\partial R_\Omega}{\partial a_l} d\Omega = 0$$

$$\int_\Omega R_\Omega \frac{\partial R_\Omega}{\partial a_l} d\Omega = 0 \quad \Rightarrow W_l = \frac{\partial R_\Omega}{\partial a_l} = \mathsf{L} N_l$$

$$\therefore \int_\Omega \mathsf{L} N_l \ R_\Omega = 0$$

$$\text{LEAST SQUARE APPROXIMATION} \neq \text{GALERKIN}$$

Galerkin Least Square approximation to PDE (GLS)

$$\int_{\Omega} \mathsf{L}N_l \; R_{\Omega} = 0$$

$$\int_{\Omega} \mathsf{L}N_l \left\{ \left(\mathsf{L}y + \sum_m a_m \mathsf{L}N_m \right) + p \right\} = 0$$

$$\int_{\Omega} \mathsf{L}N_l \sum_m a_m \; \mathsf{L}N_m \; d\Omega = -\int_{\Omega} \mathsf{L}N_l \left(\mathsf{L}y + p \right) d\Omega$$

$$K_{lm} = \int_{\Omega} \mathsf{L}N_l \; \mathsf{L}N_m \; d\Omega$$

$$f_l = -\int_{\Omega} \mathsf{L}N_l \left(\mathsf{L}y + p \right) d\Omega$$

GALERKIN LEAST SQUARE APPROXIMATION

Simultaneous approximation to the solution of the PDE in the domain and on its boundary.

$$R_\Omega = A\left(\hat{f}\right) = \mathrm{L}\left(\hat{f}\right) + \mathrm{p} = 0 \qquad \text{in } \Omega$$

$$R_\Gamma = B\left(\hat{f}\right) = \mathrm{M}\left(\hat{f}\right) + \mathrm{r} = 0 \qquad \text{in } \Gamma$$

$$f(x) \cong \hat{f}(x) = \sum_m a_m N_m(x) \qquad \text{note that } y \text{ does not exist}$$

$$\int_\Omega W_l \; R_\Omega \; d\Omega + \int_\Gamma \overline{W}_l \; R_\Gamma \; d\Gamma = 0$$

Simultaneous approximation to the solution of the PDE in the domain and on its boundary.

$$\int_\Omega W_l \; R_\Omega \; d\Omega + \int_\Gamma \overline{W}_l \; R_\Gamma \; d\Gamma = 0$$

$$\int_\Omega W_l \left(\sum_m a_m \mathrm{L} N_m + p \right) d\Omega + \int_\Gamma \overline{W}_l \left(\sum_m a_m \mathrm{M} N_m + r \right) d\Gamma = 0$$

$$\int_\Omega W_l \left(\sum_m a_m \mathrm{L} N_m \right) d\Omega + \int_\Gamma \overline{W}_l \left(\sum_m a_m \mathrm{M} N_m \right) d\Gamma = -\int_\Omega W_l \, p \; d\Omega - \int_\Gamma \overline{W}_l \; r \; d\Gamma$$

$$K_{lm} a_m = f_l$$

$$K_{lm} = \int_\Omega W_l \mathrm{L} N_m \; d\Omega + \int_\Gamma \overline{W}_l \mathrm{M} N_m \; d\Gamma$$

$$f_l = -\int_\Omega W_l \; p \; d\Omega - \int_\Gamma \overline{W}_l \; r \; d\Gamma \qquad l, m = 1, 2, \cdots, M$$

Example 3 = Example 1- but solved without psi ψ

Find $\hat{f}(x)$ solution of the following ODE

$$\frac{d^2 f}{dx^2} - f = 0 \qquad \text{in } \Omega : \{x; 0 \le x \le 1\}$$

$$f(x=0) = 0$$

$$f(x=1) = 1 \qquad \text{in } \Gamma : \{x = 0, x = 1\}$$

$$\mathsf{L}(f) = \frac{d^2 f}{dx^2} - f$$

$$p = 0$$

$$\mathsf{M}(f) + r = 0 \qquad \text{in } \Gamma$$

$$\begin{cases} \mathsf{M}(f) = f & ; \quad r = 0 \qquad \text{on } x = 0 \quad \text{DIRICHLET} \\ \mathsf{M}(f) = f & ; \quad r = -1 \qquad \text{on } x = 1 \quad \text{DIRICHLET} \end{cases}$$

Example 3 – weighted residual formulation

$$\int_{\Omega} W_l \; R_{\Omega} \; d\Omega + \int_{\Gamma} \overline{W}_l \; R_{\Gamma} \; d\Gamma = 0$$

$$\int_0^1 W_l R_{\Omega} \; dx + \left[\overline{W}_l R_{\Gamma}\right]_{x=0} + \left[\overline{W}_l R_{\Gamma}\right]_{x=1} = 0$$

$$W_l = N_l \quad \text{and} \quad \overline{W}_l = -N_l\big|_{\Gamma}$$

$$\int_0^1 N_l \left(\frac{d^2 \hat{f}}{dx^2} - \hat{f} \right) dx - \left[N_l \hat{f}\right]_{x=0} - \left[N_l \left(\hat{f} - 1\right)\right]_{x=1} = 0$$

$$\text{using} \quad N_m = x^{m-1}, \; m = 1,2,3,\cdots$$

$$K_{lm} a_m = f_l$$

$$K_{lm} = \int_0^1 N_l \left(\frac{d^2 N_m}{dx^2} - N_m \right) dx - \left[N_l N_m\right]_{x=0} - \left[N_l N_m\right]_{x=1}$$

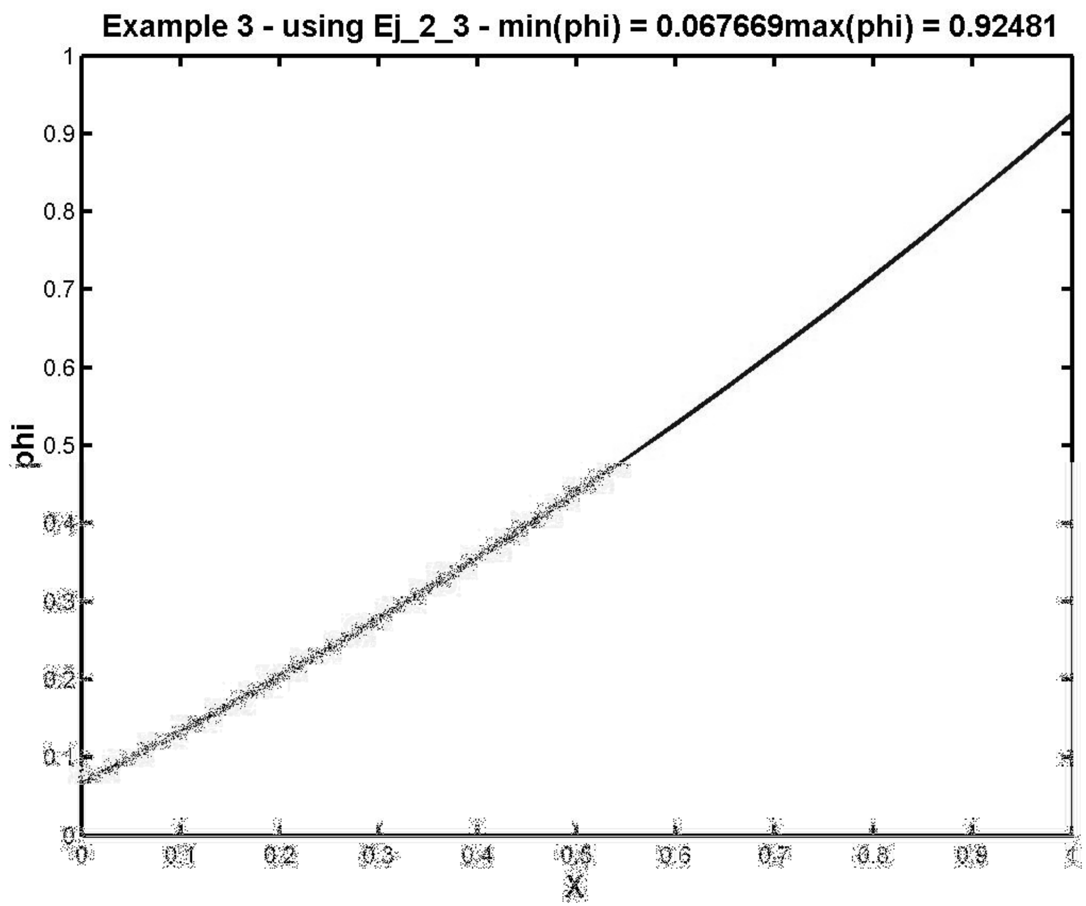
$$f_l = -\left[N_l\right]_{x=1} \qquad l, m = 1,2,\cdots,M$$

Example 3 – algebraic system

$$K\,a = f$$

$$K = -\begin{vmatrix} 3 & \frac{3}{2} & -\frac{2}{3} \\ \frac{3}{2} & \frac{4}{3} & \frac{1}{4} \\ \frac{4}{3} & \frac{5}{4} & \frac{8}{15} \end{vmatrix}$$

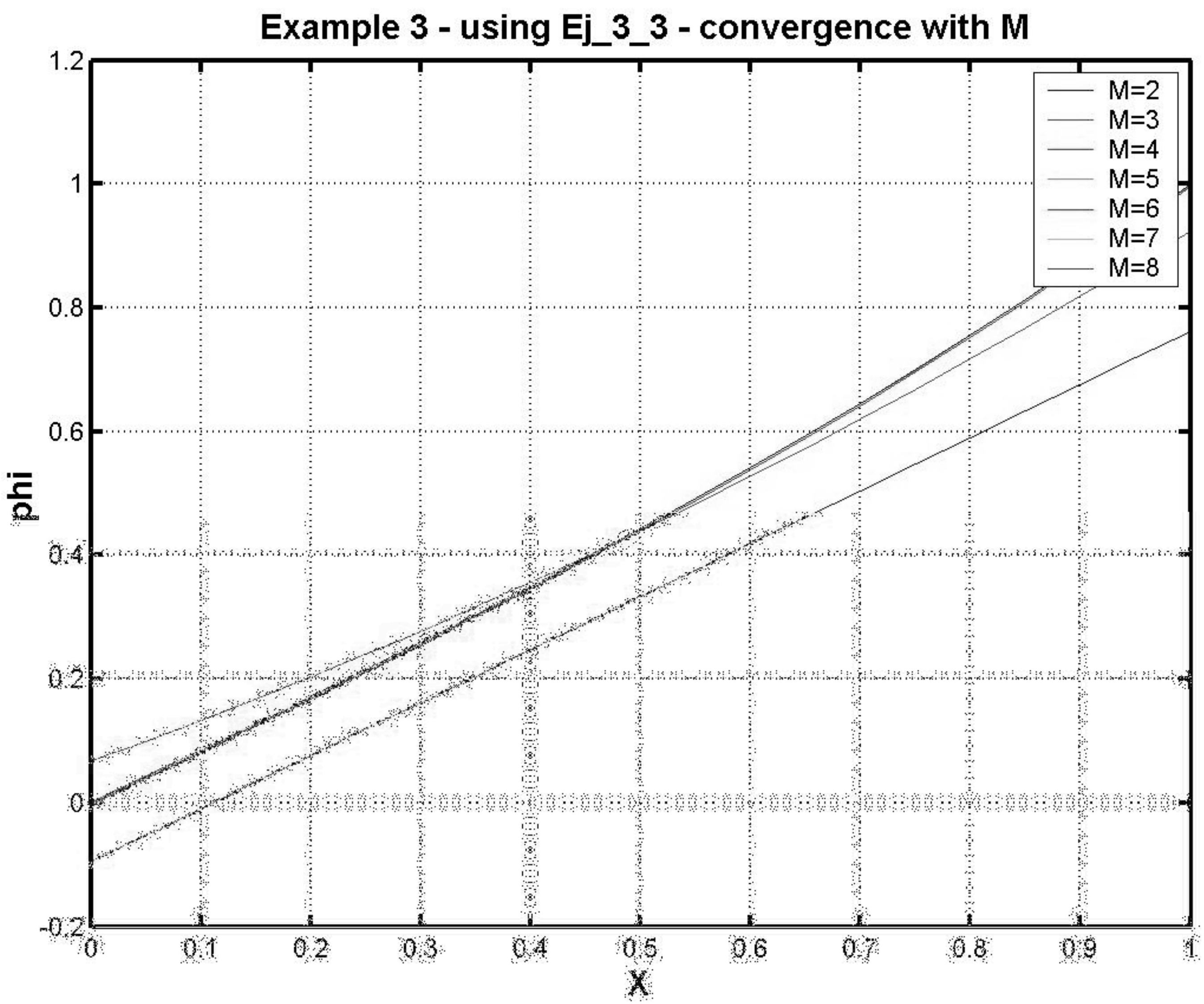
$$f = -\begin{vmatrix} 1 & 1 & 1 \end{vmatrix}^T$$



Example 3 – Results

M	**Phi (x = 0)**	**Phi (x = 1)**
2.0000e+000 -	9.5238e-002	7.6190e-001
3.0000e+000	6.7669e-002	9.2481e-001
4.0000e+000	-3.1787e-003	9.9566e-001
5.0000e+000	'M=2' 5.6585e-004	9.9940e-001
6.0000e+000	-1.3409e-005	9.9998e-001
7.0000e+000	1.4329e-006	1.0000e+000

Example 3 – algebraic system



Example 4 = Example 2- but solved without y

$$N_j = \underbrace{\left(4 - y^2\right)}_{\text{satisfy M }(f)+r=0|_{y=\pm 2}} \times N_j^*$$

$$N_1^* = 1 \quad ; \quad N_2^* = x^2 \quad ; \quad N_3^* = y^2$$

$$N_4^* = x^2 y^2 \quad ; \quad N_5^* = x^4 \quad \text{and so on}$$

$$\hat{f} = \left(4 - y^2\right)\left(a_1 + a_2 x^2 + a_3 y^2 + a_4 x^2 y^2 + a_5 x^4\right)$$

$$\int_{-3}^{3}\int_{-2}^{2} W_l \left(\frac{\partial^2 \hat{f}}{\partial \mathrm{x}^2} + \frac{\partial^2 \hat{f}}{\partial y^2} + 2 \right) dy\, dx + \int_{-2}^{2} \overline{W_l}\ \hat{f}\Big|_{x=3} dy - \int_{2}^{-2} \overline{W_l}\ \hat{f}\Big|_{x=-3} dy = 0$$

Natural boundary conditions

$$\int_{\Omega} W_l \; R_{\Omega} \; d\Omega = \int_{\Omega} W_l \left(\mathsf{L} \hat{f} + \mathrm{p} \right) d\Omega = 0$$

integration by parts allows to reduce the smoothness requirement over the functional space

$$\underbrace{\int_{\Omega} W_l \left(\mathsf{L} \hat{f} + \mathrm{p} \right) d\Omega}_{\text{strong form}} = \underbrace{\overbrace{\int_{\Omega} \mathsf{C} W_l \, \mathsf{D} \, \hat{f} \; d\Omega + \int_{\Gamma} W_l \, \mathsf{E} \, \hat{f} \; d\Gamma}^{\text{integration by parts}} + \int_{\Omega} W_l \; p d\Omega}_{\text{weak form}} = 0$$

with $\mathsf{C}, \mathsf{D}, \mathsf{E}$ linear operators of lower order than L

Natural boundary conditions

$$\underbrace{\int_{\Omega} W_l \left(\mathsf{L}\hat{f} + \mathrm{p}\right) d\Omega}_{\text{strong form}} = \underbrace{\overbrace{\int_{\Omega} \mathsf{C} W_l \, \mathsf{D}\, \hat{f}\, d\Omega + \int_{\Gamma} W_l \, \mathsf{E}\, \hat{f}\, d\Gamma}^{\text{integration by parts}} + \int_{\Omega} W_l \, p d\Omega}_{\text{weak form}} = 0$$

$$\int_{\Omega} W_l \left(\mathsf{L}\hat{f} + \mathrm{p}\right) d\Omega + \int_{\Gamma} \overline{W_l} \left(\mathsf{M}\, \hat{f} + \mathrm{r}\right) d\Gamma =$$

$$= \int_{\Omega} \mathsf{C} W_l \, \mathsf{D}\, \hat{f}\, d\Omega + \int_{\Omega} W_l \, p d\Omega + \int_{\Gamma} W_l \, \mathsf{E}\, \hat{f}\, d\Gamma + \int_{\Gamma} \overline{W_l} \left(\mathsf{M}\, \hat{f} + \mathrm{r}\right) d\Gamma = 0$$

combining the last two boundary integrals, is it possible to cancel out ?

This is only possible for certain selections of $\overline{W_l}$ and boundary conditions

Example 5 – natural boundary conditions

Find $\hat{f}(x)$ solution of the following ODE

$$\frac{\mathrm{d}^2 f}{\mathrm{dx}^2} - f = 0 \qquad \text{in } \Omega : \{\mathrm{x}; 0 \le \mathrm{x} \le 1\}$$

$$f(x=0) = 0$$

$$\frac{df}{dx}(x=1) = 20 \qquad \text{in } \Gamma : \{\mathrm{x}=0, \mathrm{x}=1\}$$

using $N_m = x^m$

$$\int_0^1 W_l R_\Omega \, dx + \left[\overline{W}_l R_\Gamma\right]_{x=0} + \left[\overline{W}_l R_\Gamma\right]_{x=1} = 0$$

$$\int_0^1 W_l \left(\frac{d^2 \hat{f}}{dx^2} - \hat{f} \right) dx + \left[\overline{W}_l \hat{f}\right]_{x=0} + \left[\overline{W}_l \left(\frac{d\hat{f}}{dx} - 20 \right)\right]_{x=1} = 0$$

$$-\int_0^1 \frac{dW_l}{dx} \frac{d\hat{f}}{dx} dx - \int_0^1 W_l \, \hat{f} \, dx + \left[W \frac{d\hat{f}}{dx} \right]_{x=1} - \left[W \frac{d\hat{f}}{dx} \right]_{x=0} +$$

$$\underbrace{\left[\overline{W}_l \hat{f}\right]_{x=0}}_{=0\,(\mathrm{DIRICHLET})} + \left[\overline{W}_l \left(\frac{df}{dx} - 20 \right)\right]_{x=1} = 0$$

Example 5 – natural boundary conditions

Choosing $\overline{W}_l\Big|_{x=1} = -W_l\Big|_{x=1}$ & $W_l\Big|_{x=0} = 0$

$$-\int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx - \int_0^1 W_l\,\hat{f}\,dx + \left[W_l\,\frac{d\hat{f}}{dx}\right]_{x=1} - \left[W_l\,\frac{d\hat{f}}{dx}\right]_{x=0} +$$

$$\underbrace{\left[\overline{W_l}\hat{f}\right]_{x=0}}_{=0\text{ (DIRICHLET)}} + \left[\overline{W_l}\left(\frac{d\hat{f}}{dx} - 20\right)\right]_{x=1} = 0$$

$$\Rightarrow \qquad \int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx + \int_0^1 W_l\,\hat{f}\,dx = \left[20\,W_l\right]_{x=1}$$

there is no need to compute the $\left.\dfrac{d\hat{f}}{dx}\right|_{\Gamma}$

Gradient boundary conditions are natural for such weak formulation

Moreover, if $\left.\dfrac{d\hat{f}}{dx}\right|_{\Gamma} = 0 \Rightarrow rhs = 0$

Example 5 – natural boundary conditions

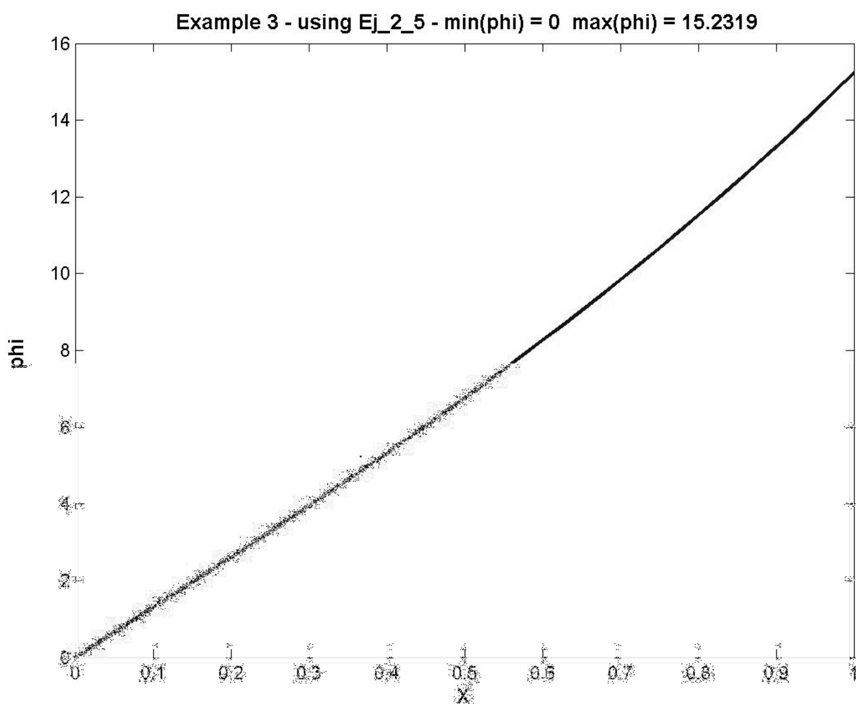
$$\int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx + \int_0^1 W_l\ \hat{f}\ dx = \left[20\,W_l\right]_{x=1}$$

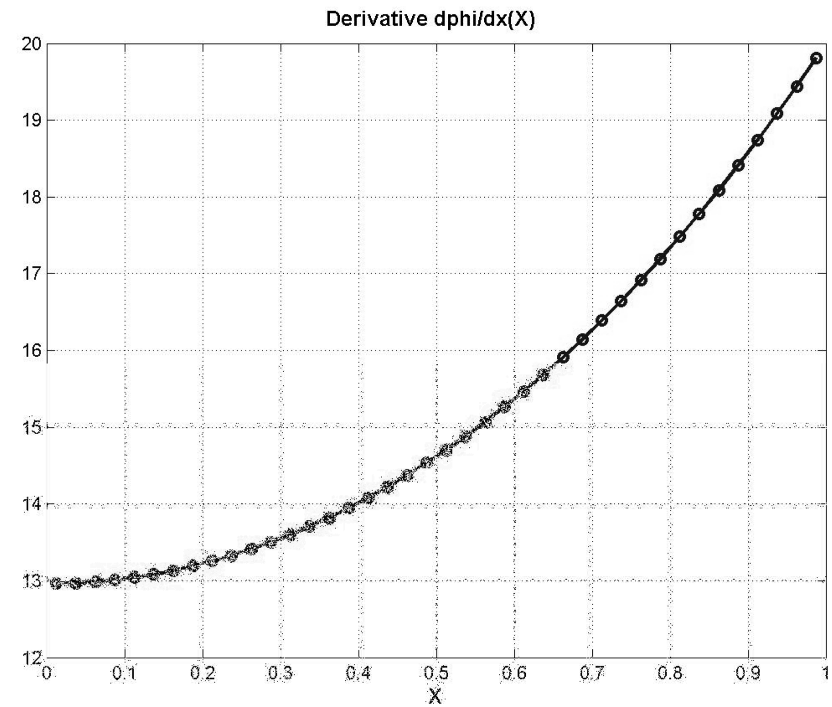
$$\Rightarrow \quad K\ a\ = \ f$$

$$K_{lm} = \int_0^1 \frac{dW_l}{dx}\frac{dN_m}{dx}dx + \int_0^1 W_l\ N_m dx$$

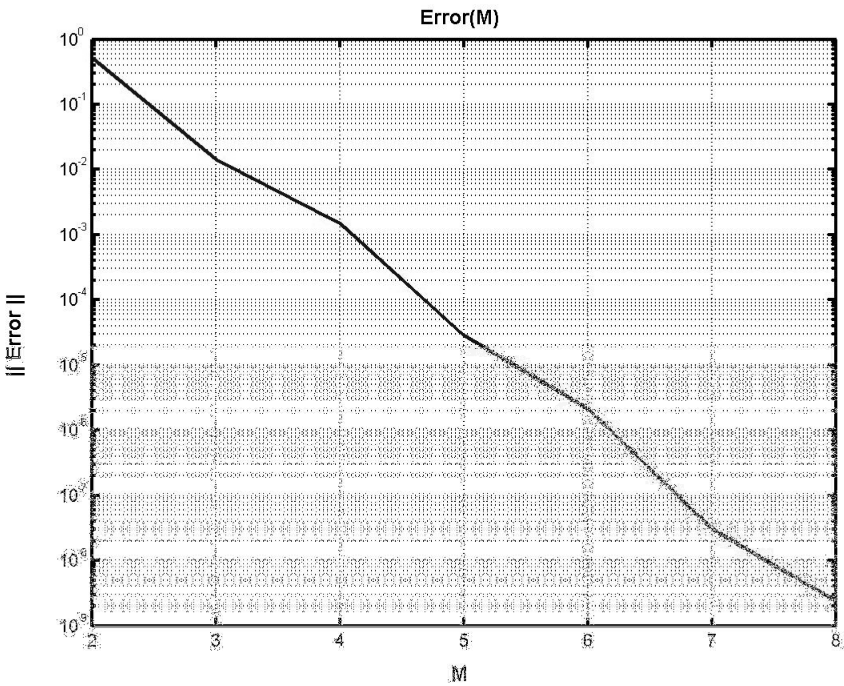
$$f_l \quad = \ 20\,W_l\Big|_{x=1}$$

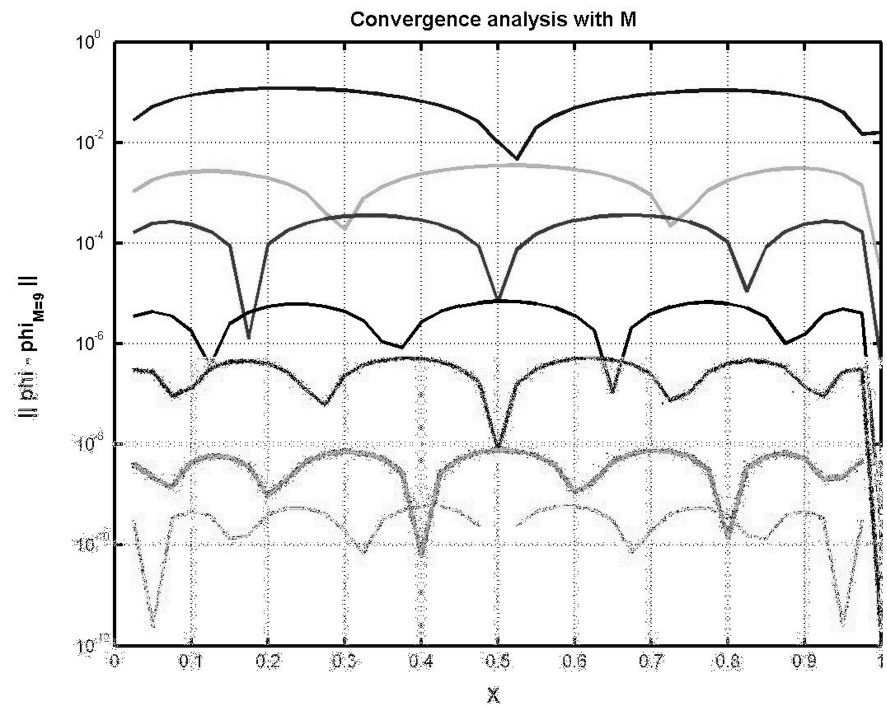
Example 5 – Results (routine Ej_2_5.m)





Example 5 – Results (routine Ej_2_5.m)





Natural boundary condition for heat equation

Find $\hat{f}(x)$ solution of the following PDE

$$\frac{\partial}{\partial \mathrm{x}}\left(k\frac{\partial f}{\partial \mathrm{x}}\right)+\frac{\partial}{\partial \mathrm{y}}\left(k\frac{\partial f}{\partial \mathrm{y}}\right)+Q=0 \qquad \text{in } \Omega:\left\{\mathrm{x};\, x\in \Re^2\right\}$$

$$f=\bar{f} \qquad \text{on } \Gamma_f$$

$$k\frac{\partial f}{\partial h}=-\bar{q} \qquad \text{on } \Gamma_q$$

$$\hat{f}=y+\sum_m a_m N_m$$

$$y=\bar{f} \qquad \text{on } \Gamma_f$$

$$N_m=0 \qquad \text{on } \Gamma_f$$

$$\int_\Omega W_l\left(\frac{\partial}{\partial \mathrm{x}}\left(k\frac{\partial \hat{f}}{\partial \mathrm{x}}\right)+\frac{\partial}{\partial \mathrm{y}}\left(k\frac{\partial \hat{f}}{\partial \mathrm{y}}\right)+Q\right)d\Omega+\int_{\Gamma_q}\overline{W_l}\left(k\frac{\partial \hat{f}}{\partial h}+\bar{q}\right)d\Gamma=0$$

Natural boundary condition for heat equation

$$\int_\Omega W_l\left(\frac{\partial}{\partial x}\left(k\frac{\partial\hat{f}}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial\hat{f}}{\partial y}\right)+Q\right)d\Omega+\int_{\Gamma_q}\overline{W_l}\left(k\frac{\partial\hat{f}}{\partial h}+\bar{q}\right)d\Gamma=0$$

Applying the Green's lemma

$$\int_\Omega\left\{-\left(\frac{\partial W_l}{\partial x}\left(k\frac{\partial\hat{f}}{\partial x}\right)+\frac{\partial W_l}{\partial y}\left(k\frac{\partial\hat{f}}{\partial y}\right)\right)+W_lQ\right\}d\Omega+$$

$$+\int_{\Gamma_f\cup\Gamma_q}W_l\left(k\frac{\partial\hat{f}}{\partial h}+\bar{q}\right)d\Gamma+\int_{\Gamma_q}\overline{W_l}\left(k\frac{\partial\hat{f}}{\partial h}+\bar{q}\right)d\Gamma=0$$

choosing the weighted functions so that

$$W_l=0 \quad \text{on} \quad \Gamma_f$$

$$\overline{W_l}=-W_l \quad \text{on} \quad \Gamma_q$$

$$\int_\Omega\left(\frac{\partial W_l}{\partial x}\left(k\frac{\partial\hat{f}}{\partial x}\right)+\frac{\partial W_l}{\partial y}\left(k\frac{\partial\hat{f}}{\partial y}\right)-W_lQ\right)d\Omega+\int_{\Gamma_q}W_l\,\bar{q}\,d\Gamma=0$$

Natural boundary condition for heat equation

$$K\ a = f$$

$$K_{lm} = \int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(k \frac{\partial N_m}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(k \frac{\partial N_m}{\partial y} \right) \right) d\Omega$$

$$f_l = \int_{\Omega} W_l Q\ d\Omega + \int_{\Gamma_q} W_l \bar{q}\ d\Gamma - \int_{\Omega} \left(\frac{\partial W_l}{\partial x} \left(k \frac{\partial y}{\partial x} \right) + \frac{\partial W_l}{\partial y} \left(k \frac{\partial y}{\partial y} \right) \right) d\Omega = 0$$

$$1 \le l, m \le M$$

using Galerkin approximation ($W_l = N_l$)

$\therefore \quad K_{lm} = K_{ml}$ symmetry

$k \dfrac{\partial f}{\partial h} = -\bar{q}$ on Γ_q $\Rightarrow$ there is no phi gradient at boundary terms

$\therefore \quad k \dfrac{\partial f}{\partial h} = -\bar{q}$ is the natural BC for heat equation

Example 6 – natural BC for heat equation

Find $\hat{f}(x)$ solution of the following PDE in 2D

$$\frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial f}{\partial \mathrm{x}}\right)+\frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial f}{\partial \mathrm{y}}\right)=0 \qquad \text{in } \Omega:\{\mathrm{x},\mathrm{y};x,y\in(-1,1)\times(-1,1)\}$$

$$f=0 \quad \text{on } \Gamma_f:\{\mathrm{x},\mathrm{y};y=\pm 1\}$$

$$\frac{\partial f}{\partial h}=\cos\left(\frac{p\ y}{2}\right) \quad \text{on } \Gamma_q:\{\mathrm{x},\mathrm{y};x=\pm 1\}$$

$$N_j=\underbrace{\left(1-y^2\right)}_{\text{satisfy } \mathrm{M}\ (f)+r=0|_{y=\pm 1}}\times N_j^*$$

$$N_1^*=1 \quad ; \quad N_2^*=x^2 \quad ; \quad N_3^*=y^2$$

$$N_4^*=x^2y^2 \quad ; \quad N_5^*=x^4 \quad \text{and so on}$$

$$\hat{f}=\left(1-y^2\right)\left(a_1+a_2x^2+a_3y^2+a_4x^2y^2+a_5x^4\right)$$

immediately satisfies the BC @ Γ_f

$$\int_{-1}^{1}\int_{-1}^{1}W_l\left(\frac{\partial^2\hat{f}}{\partial \mathrm{x}^2}+\frac{\partial^2\hat{f}}{\partial y^2}\right)dy\,dx+\int_{\Gamma_q}\overline{W_l}\left(\frac{\partial\hat{f}}{\partial h}-\cos\left(\frac{p\ y}{2}\right)\right)d\Gamma=0$$

Example 6 – weak form

$$\int_{-1}^{1}\int_{-1}^{1} W_l \left(\frac{\partial^2 \hat{f}}{\partial \mathrm{x}^2} + \frac{\partial^2 \hat{f}}{\partial y^2} \right) dy\, dx + \int_{\Gamma_q} \overline{W_l} \left(\frac{\partial \hat{f}}{\partial h} - \cos\left(\frac{p\ y}{2} \right) \right) d\Gamma = 0$$

$$\int_{-1}^{1}\int_{-1}^{1} \left(\frac{\partial W_l}{\partial \mathrm{x}} \frac{\partial \hat{f}}{\partial \mathrm{x}} + \frac{\partial W_l}{\partial \mathrm{y}} \frac{\partial \hat{f}}{\partial y} \right) dy\, dx -$$

$$- \int_{\Gamma_q \cup \Gamma_f} W_l \frac{\partial \hat{f}}{\partial h}\, d\Gamma - \int_{\Gamma_q} \overline{W_l} \left(\frac{\partial \hat{f}}{\partial h} - \cos\left(\frac{p\ y}{2} \right) \right) d\Gamma = 0$$

Example 6 – weak form

$$W_l = N_l \quad \text{(Galerkin)}$$

$$\Rightarrow \int_{\Gamma_f} W_l \frac{\partial \hat{f}}{\partial h}\, d\Gamma = 0 \quad \text{because } N_l\Big|_{\Gamma_f} = 0$$

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial N_l}{\partial \mathrm{x}}\frac{\partial \hat{f}}{\partial \mathrm{x}} + \frac{\partial N_l}{\partial \mathrm{y}}\frac{\partial \hat{f}}{\partial y}\right) dy\, dx -$$

$$- \int_{\Gamma_q} N_l \frac{\partial \hat{f}}{\partial h}\, d\Gamma - \int_{\Gamma_q} \overline{W_l}\left(\frac{\partial \hat{f}}{\partial h} - \cos\left(\frac{p\ \ y}{2}\right)\right) d\Gamma = 0$$

$$\text{taking } \overline{W_l}\Big|_{\Gamma_q} = -\overline{N_l}\Big|_{\Gamma_q} \quad \text{cancel out the derivative term @ } \Gamma_q$$

$$\therefore \quad \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial N_l}{\partial \mathrm{x}}\frac{\partial \hat{f}}{\partial \mathrm{x}} + \frac{\partial N_l}{\partial \mathrm{y}}\frac{\partial \hat{f}}{\partial y}\right) dy\, dx - \int_{\Gamma_q} N_l \cos\left(\frac{p\ \ y}{2}\right)\left(\frac{\partial \hat{f}}{\partial h} - \right) d\Gamma = 0$$

Example 6 – linear system

$$\therefore \quad K_{lm} = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial N_l}{\partial \mathrm{x}}\frac{\partial \hat{f}}{\partial \mathrm{x}} + \frac{\partial N_l}{\partial \mathrm{y}}\frac{\partial \hat{f}}{\partial y}\right) dy\, dx$$

$$f_l = \int_{-1}^{1} N_l\Big|_{x=1} \cos\left(\frac{p\ \ y}{2}\right) dy - \int_{1}^{-1} N_l\Big|_{x=-1} \cos\left(\frac{p\ \ y}{2}\right) dy$$

equivalent to

$$K_{lm} = \int_{0}^{1}\int_{0}^{1}\left(\frac{\partial N_l}{\partial \mathrm{x}}\frac{\partial \hat{f}}{\partial \mathrm{x}} + \frac{\partial N_l}{\partial \mathrm{y}}\frac{\partial \hat{f}}{\partial y}\right) dy\, dx$$

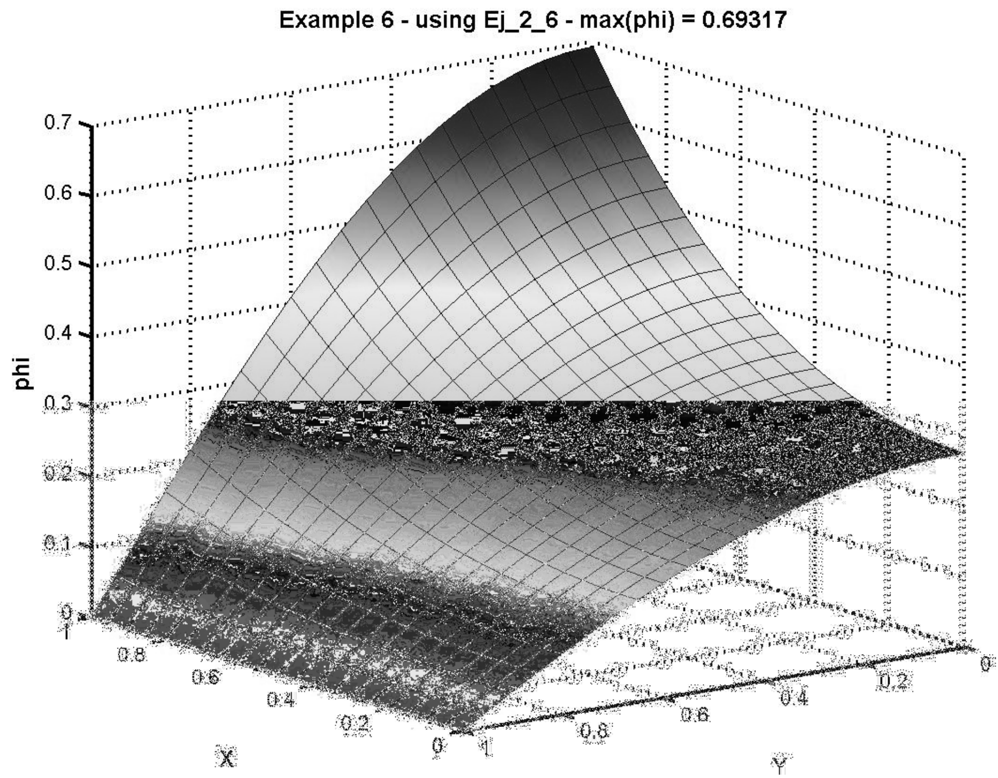
2D integral

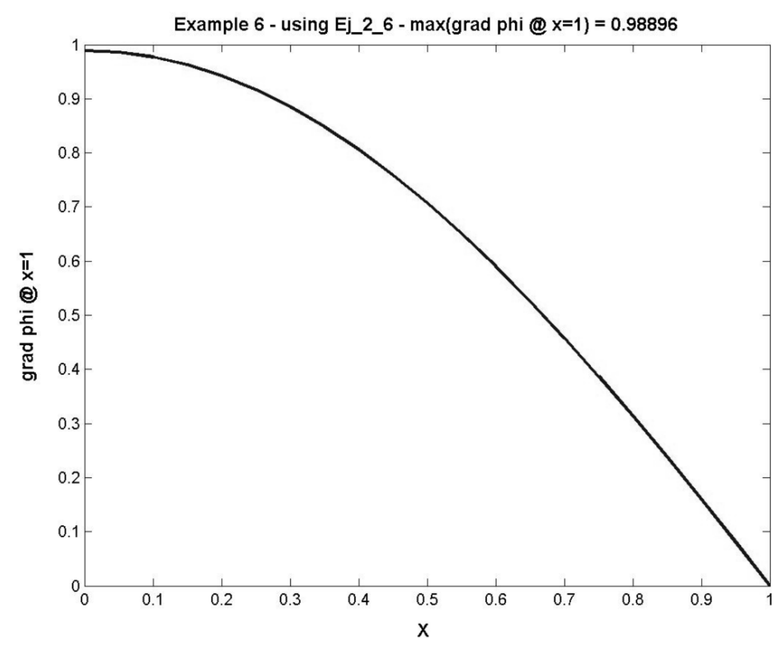
$$f_l = \int_{0}^{1} N_l\Big|_{x=1} \cos\left(\frac{p\ \ y}{2}\right) dy$$

1D integral

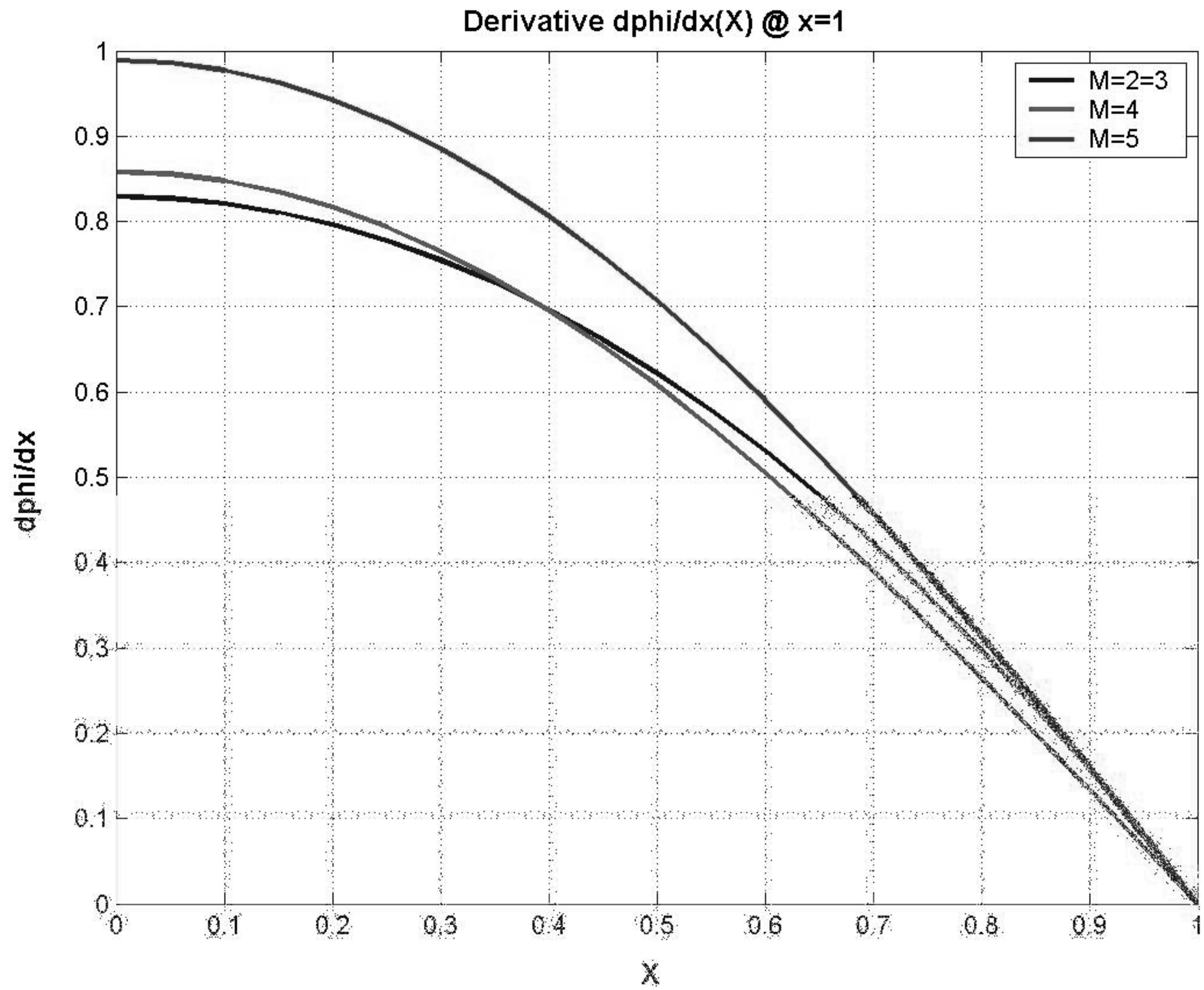
```
f = zeros(M,1);
K = zeros(M,M);
for ll=1:M
   f(ll) =
gauss_integration('ffun_Ej_2_6_rhs',1,c,d);
   for mm=1:M
      K(ll,mm) =
gauss_integration('ffun_Ej_2_6_lhs',2,a,b,c,d);
   end
end
```

Example 6 – Results (routine Ej_2_6.m)





Example 6 – Results (routine Ej_2_6.m)



Example 6 – Results (routine Ej_2_6.m)

K = […

1.3333e+000	**4.4444e-001**	**2.6667e-001**	**8.8889e-002**	**2.6667e-001**
4.4444e-001	**9.7778e-001**	**8.8889e-002**	**1.5492e-001**	**1.0438e+000**
2.6667e-001	**8.8889e-002**	**4.1905e-001**	**1.3968e-001**	**5.3333e-002**
8.8889e-002	**1.5492e-001**	**1.3968e-001**	**1.1767e-001**	**1.6000e-001**
2.6667e-001	**1.0438e+000**	**5.3333e-002**	**1.6000e-001**	**1.3672e+000]**

f’ = [5.1602e-001 5.1602e-001 7.0480e-002 7.0480e-002 5.1602e-001]

a’ = [2.7631e-001 3.3925e-001 -5.8747e-002 -9.2206e-002 7.7616e-002]

Boundary solution methods

[1] $\psi(x)$ to satisfy BC

$$f(x) \cong \hat{f}(x) = \psi(x) + \sum_m a_m N_m(x)$$

$$\text{on } \Gamma \quad \begin{cases} \mathrm{M}(\psi) = -r \\ \mathrm{M}(N_m) = 0 \quad ; \quad m = 1, 2, \cdots \end{cases}$$

Weighted Residual excusively over the domain

$$\int_\Omega W_l \, R_\Omega \, d\Omega = \int_\Omega W_l \left(\mathrm{L}\psi + \sum_m a_m \mathrm{L} N_m + p \right) d\Omega = 0$$

[2] give up $\psi(x)$ to satisfy BC and add a boundary residual term in the formulation

$$f(x) \cong \hat{f}(x) = \sum_m a_m N_m(x)$$

$$R_\Omega = A(f) = \mathrm{L}(f) + \mathrm{p} = 0 \qquad \text{in } \Omega$$

$$R_\Gamma = B(f) = \mathrm{M}(f) + \mathrm{r} = 0 \qquad \text{in } \Gamma$$

$$\int_\Omega W_l \, R_\Omega \, d\Omega + \int_\Omega \overline{W}_l \, R_\Gamma \, d\Gamma = 0$$

Boundary solution methods

$$[1] \quad \Rightarrow \quad \int_{\Omega} W_l \; R_{\Omega} \; d\Omega = 0$$

$$[2] \quad \Rightarrow \quad \int_{\Omega} W_l \; R_{\Omega} \; d\Omega + \int_{\Omega} \overline{W}_l \; R_{\Gamma} \; d\Gamma = 0$$

why don't propose to solve exclusively the boundary

including for the approximation the solution inside the domain ?

Probably it is difficult to know the analytical solution in the domain

Only for simple (linear) differential operators

But, the dimension of the problem is reduced in one (3D $\rightarrow$ 2D)

$$[3] \quad \Rightarrow \quad \int_{\Omega} \overline{W}_l \; R_{\Gamma} \; d\Gamma = 0$$

choosing N_m such that $A(N_m) = 0 \quad \therefore \; R_{\Omega} = A(f) = \sum_m a_m A(N_m) = 0$

Boundary solution methods to Laplace equation

Any analytic function of the complex variable $z = x + i\,y$

$$f(z) = u + i\,v$$

with $u, v \in \Re$ satisfy

$$\frac{\partial^2 f}{\partial x^2} = f'' \quad , \quad \frac{\partial^2 f}{\partial y^2} = i^2 f'' = -f'' \quad , \quad f'' = \frac{d^2 f}{dz^2}$$

$$\therefore \nabla^2 f = \nabla^2 u + i \nabla^2 v = 0 \quad \Rightarrow \quad \nabla^2 u = \nabla^2 v = 0$$

$\therefore$ using $f(z) = z^n$

$$n = 1 \quad , \quad u = x \quad , \quad v = y$$

$$n = 2 \quad , \quad u = x^2 - y^2 \quad , \quad v = 2xy$$

$$n = 3 \quad , \quad u = x^3 - 3xy^2 \quad , \quad v = 3x^2 y - y^3$$

$$n = 4 \quad , \quad u = x^4 - 6x^2 y^2 + y^4 \quad , \quad v = 4x^3 y - 4xy^3$$

Example 7- Torsion problem by BSM

Find $\hat{f}(x)$ solution of the following PDE

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -2 \qquad \text{in } \Omega : \{(x, y); -3 \le x \le 3; -2 \le y \le 2\}$$

$$f(x=-3) = f(x=3) = 0$$

$$f(y=-2) = f(y=2) = 0$$

equivalent to solve

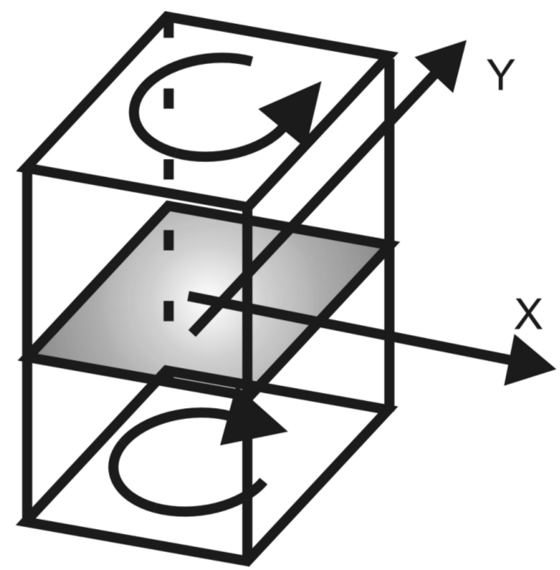
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -2 \qquad \text{in } \Omega : \{(x, y); 0 \le x \le 3; 0 \le y \le 2\}$$

$$f(x=3) = 0$$

$$f(y=2) = 0$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = 0$$



Boundary solution methods - Example

Changing variables

$$f = q - \frac{1}{2}\left(x^2 + y^2\right)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -2 \quad \Rightarrow \quad \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = 0$$

Approximating q

by symmetric functions respect to x and y

$$q \approx \hat{q} = a_1 \underbrace{1}_{N_1} + a_2 \underbrace{\left(x^2 - y^2\right)}_{N_2} + a_3 \underbrace{\left(x^4 - 6x^2y^2 + y^4\right)}_{N_3}$$

$$\frac{\partial^2 \hat{q}}{\partial x^2} + \frac{\partial^2 \hat{q}}{\partial y^2} = 0 \quad , \quad \forall (x, y) \in \Omega$$

$$\int_\Omega \overline{W}_l \; R_\Gamma \; d\Gamma = 0$$

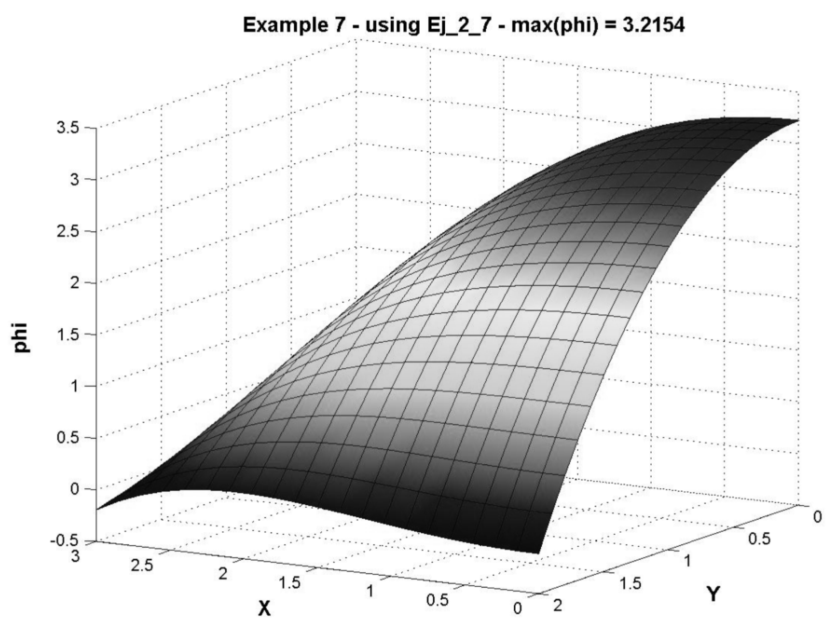
$$\Rightarrow \int_\Omega N_l\big|_{x=3} \left\{ \hat{q}\big|_{x=3} - \frac{1}{2}\left(9 + y^2\right) \right\} dy + \int_\Omega N_l\big|_{y=2} \left\{ \hat{q}\big|_{y=2} - \frac{1}{2}\left(x^2 + 4\right) \right\} dx = 0$$

Boundary solution methods - Example

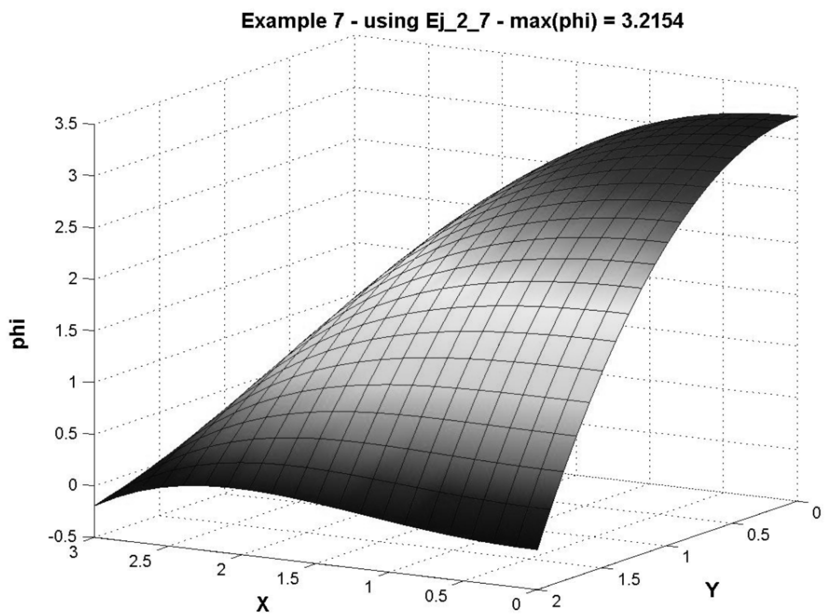
```
f = zeros(M,1);
K = zeros(M,M);
for ll=1:M
   f1 = gauss_integration('ffun_Ej_2_7_rhs_f1',1,c,d);
   f2 = gauss_integration('ffun_Ej_2_7_rhs_f2',1,a,b);
   f(ll) = -(f1+f2);
   for mm=1:M
      K1 = gauss_integration('ffun_Ej_2_7_lhs_f1',1,c,d);
      K2 = gauss_integration('ffun_Ej_2_7_lhs_f2',1,a,b);
      K(ll,mm) = K1+K2;
   end
end
```

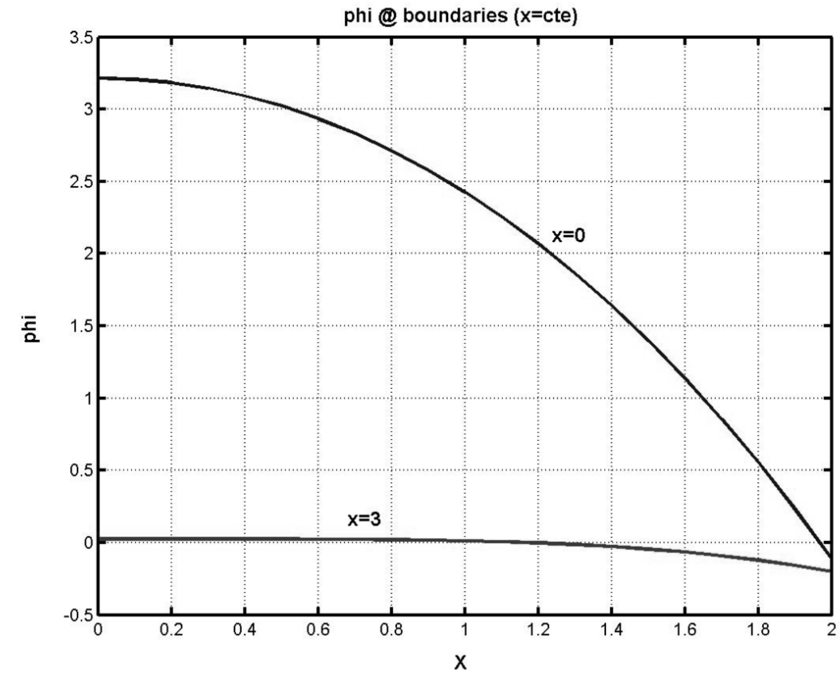
$$\int_{\Omega} N_l\big|_{x=3}\left\{\hat{q}\big|_{x=3} - \frac{1}{2}\left(9+y^2\right)\right\} dy +$$

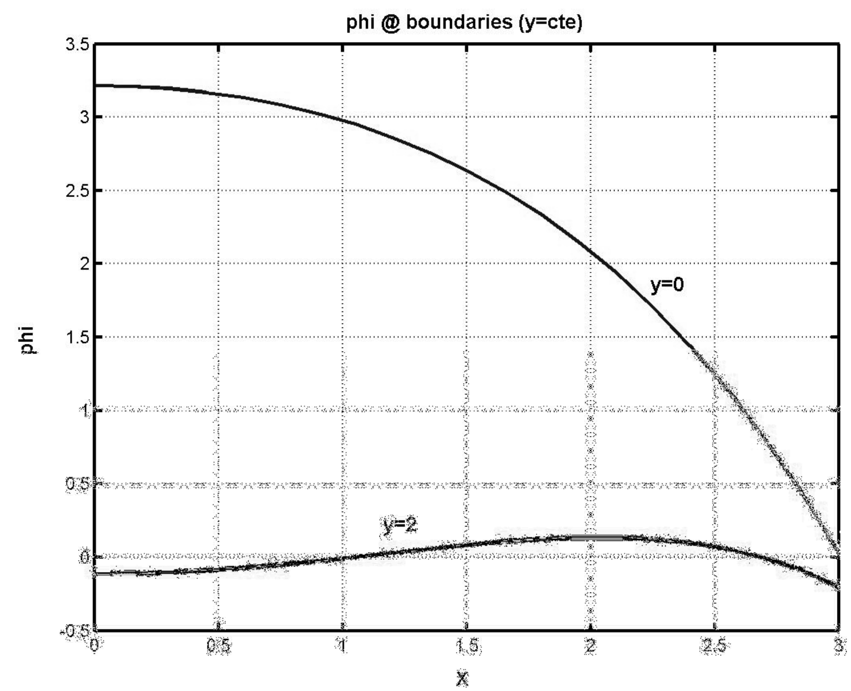
$$\int_{\Omega} N_l\big|_{y=2}\left\{\hat{q}\big|_{y=2} - \frac{1}{2}\left(x^2+4\right)\right\} dx = 0$$



Boundary solution methods - Example







How the solution fits the right Dirichlet boundary values ?

Systems of differential equations

- Simultaneous resolution of PDEs coming from different applications
- 2D, axisymetric or 3D linear elasticity
- 1D beam equations
- 1D Gas dynamics
- Fluid mechanics in general
- Maxwell equations for electromagnetism
- Scalar 2nd order PDEs transformed to a system of 1st order PDE

Systems of differential equations

$$\underline{f}^T = (f_1, f_2, f_3, \ldots)$$

$$\begin{cases} A_1(f) = 0 \\ A_2(f) = 0 \quad \text{in } \Omega \\ \vdots \qquad\qquad \vdots \end{cases} \quad ; \quad A(f) = \begin{bmatrix} A_1(f) \\ A_2(f) \\ \vdots \end{bmatrix} = 0 \quad \text{in } \Omega$$

$$\begin{cases} B_1(f) = 0 \\ B_2(f) = 0 \quad \text{in } \Gamma \\ \vdots \qquad\qquad \vdots \end{cases} \quad ; \quad B(f) = \begin{bmatrix} B_1(f) \\ B_2(f) \\ \vdots \end{bmatrix} = 0 \quad \text{in } \Gamma$$

$$f_1(x) \cong \hat{f}_1(x) = y_1(x) + \sum_m a_{m,1} N_{m,1}(x)$$

$$f_2(x) \cong \hat{f}_2(x) = y_2(x) + \sum_m a_{m,2} N_{m,2}(x)$$

$$\vdots$$

$$\underline{f(x)} \cong \underline{\hat{f}(x)} = \underline{y(x)} + \sum_m \underline{a_m} \, \underline{\underline{N_m}}(x)$$

$$\underline{y(x)}^T = (y_1, y_2, y_3, \ldots) \quad ; \quad \underline{a_m}^T = (a_{m,1}, a_{m,2}, a_{m,3}, \ldots) \quad ; \quad \underline{\underline{N_m}} = \begin{bmatrix} N_{m,1} & & & 0 \\ & N_{m,2} & & \\ & & N_{m,3} & \\ 0 & & & \ddots \end{bmatrix}$$

Systems of differential equations

$$\int_\Omega W_{l,1}\; A_1\left(\hat{f}\right) d\Omega + \int_\Gamma \overline{W}_{l,1}\; B_1\left(\hat{f}\right) d\Gamma = 0$$

$$\int_\Omega W_{l,2}\; A_2\left(\hat{f}\right) d\Omega + \int_\Gamma \overline{W}_{l,2}\; B_2\left(\hat{f}\right) d\Gamma = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\underline{\underline{W_l}} = \begin{bmatrix} W_{l,1} & & & 0 \\ & W_{l,2} & & \\ & & W_{l,3} & \\ 0 & & & \ddots \end{bmatrix} \quad ; \quad \underline{\underline{\overline{W}_l}} = \begin{bmatrix} \overline{W}_{l,1} & & & 0 \\ & \overline{W}_{l,2} & & \\ & & \overline{W}_{l,3} & \\ 0 & & & \ddots \end{bmatrix}$$

$$\int_\Omega \underline{W_l}\; \underline{A}\left(\underline{\hat{f}}\right) d\Omega + \int_\Gamma \underline{\overline{W}_l}\; \underline{B}\left(\underline{\hat{f}}\right) d\Gamma = 0$$

Example 8 : SoDE – 2nd order to 1st order

- Scalar heat equation (Poisson equation)
- In 1D
- 2nd order ODE (steady + 1D)
- Transformed to 1st order system of ODEs

$$\begin{cases} q + k\dfrac{df}{dx} = 0 \\ \dfrac{dq}{dx} - Q = 0 \end{cases} \quad ; \quad A(f) = \begin{bmatrix} q + k\dfrac{df}{dx} \\ \dfrac{dq}{dx} - Q \end{bmatrix} = 0 \quad \text{in } \Omega$$

$$\underline{f}^T = (q, f)$$

$$\mathsf{L}\,\underline{f} + \underline{p} = \begin{bmatrix} 1 & k\dfrac{d}{dx} \\ \dfrac{d}{dx} & 0 \end{bmatrix} \underline{f} + \begin{bmatrix} 0 \\ -Q \end{bmatrix} = 0$$

Example 8 : SoDE – 2nd order to 1st order

- Scalar heat equation (Poisson equation)
- In 1D
- 2nd order ODE (steady + 1D)
- Transformed to 1st order system of ODEs

$$\Omega : \{x; 0 \le x \le 1\} \quad ; \quad k = 1 \quad ; \quad Q = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

$$f = 0 \quad @\, x = 0 \quad ; \quad q = 0 \quad @\, x = 1$$

$$\hat{q} = y_1 + \sum_m a_{m,1} N_{m,1}$$

$$\hat{f} = y_2 + \sum_m a_{m,2} N_{m,2}$$

$$y_1 = y_2 = 0 \quad ; \quad N_{m,1} = 0 \quad @\, x = 1 \quad ; \quad N_{m,2} = 0 \quad @\, x = 0$$

Example 8 : SoDE – 2nd order to 1st order

$$N_{m,1} = x^{m-1}(1-x) \qquad ; \qquad N_{m,2} = x^m$$

$$\int_0^1 \underline{W_l}\left(\mathsf{L}\,\underline{\hat{f}} + \underline{p}\right) dx = 0$$

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1M} \\ K_{21} & K_{22} & \cdots & K_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ K_{M1} & K_{M2} & \cdots & K_{MM} \end{bmatrix} \quad ; \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_M \end{bmatrix}$$

$$K_{lm} = \int_0^1 \underline{W_l}\,\mathsf{L}\,\underline{N_m}\,dx \qquad ; \qquad f_l = -\int_0^1 \underline{W_l}\;\underline{p}\,dx$$

Galerkin type weighting $\underline{W_l} = \underline{N_l}$

$$K_{lm} = \begin{bmatrix} \int_0^1 N_{l,1} N_{m,1}\,dx & \int_0^1 N_{l,1} \frac{dN_{m,2}}{dx}\,dx \\ \int_0^1 N_{l,2} \frac{dN_{m,1}}{dx}\,dx & 0 \end{bmatrix}$$

Example 8 : SoDE – 2nd order to 1st order

```
f = zeros(M,ndf); K = zeros(M,M,ndf,ndf);

coef.ncol = ndf;
for ll=1:M
    coef.ncol = ndf;
    f(ll,1:ndf) = gauss_integration('ffun_Ej_2_8_rhs',0,1);
    for mm=1:M
        coef.ncol = ndf*ndf;
        vaux = gauss_integration('ffun_Ej_2_8_lhs',0,1);
        K(ll,mm,1:ndf,1:ndf) = reshape(vaux,ndf,ndf);
    end
end
```

Assembling

```
K2 = zeros(ndf*M,ndf*M); f2 = zeros(ndf*M,1);
for ll=1:M
    ll2 = (ll-1)*ndf+(1:ndf);
    f2(ll2) = reshape(f(ll,1:ndf),ndf,1);
    for mm=1:M
        mm2 = (mm-1)*ndf+(1:ndf);
        K2(ll2,mm2) = reshape(K(ll,mm,1:ndf,1:ndf),ndf,ndf);
    end
end

a = K2\f2;
```

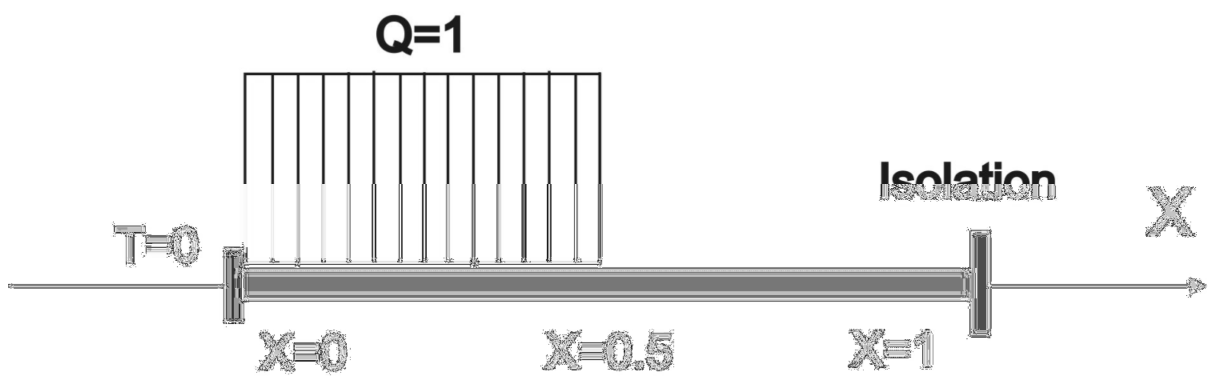
Reordering

Example 8 : SoDE – 2nd order to 1st order

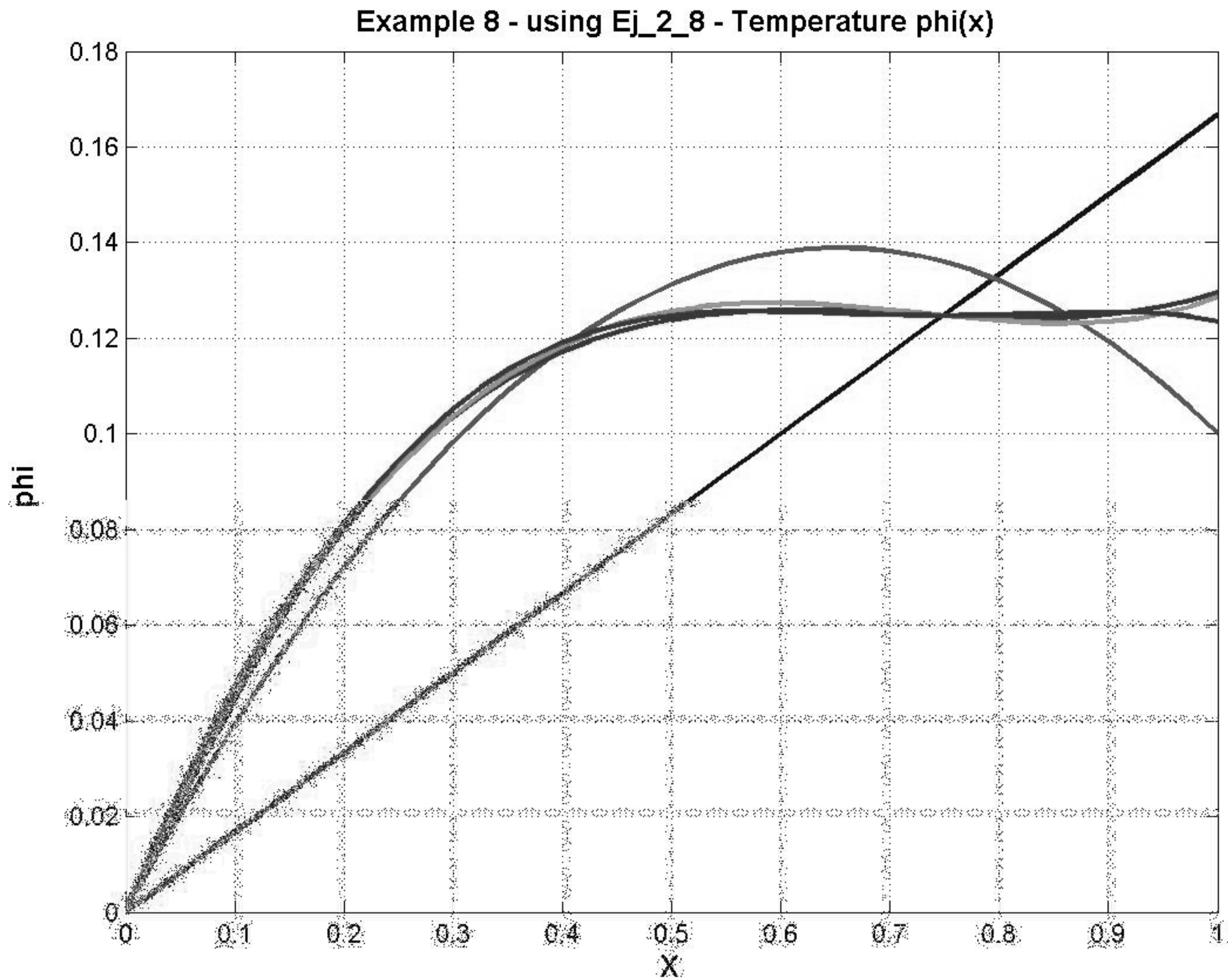
$$\frac{d}{dx}\left(k\,\frac{df}{dx}\right) - Q = 0$$

$$\Omega : \{x; 0 \le x \le 1\} \quad ; \quad k = 1 \quad ; \quad Q = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}$$

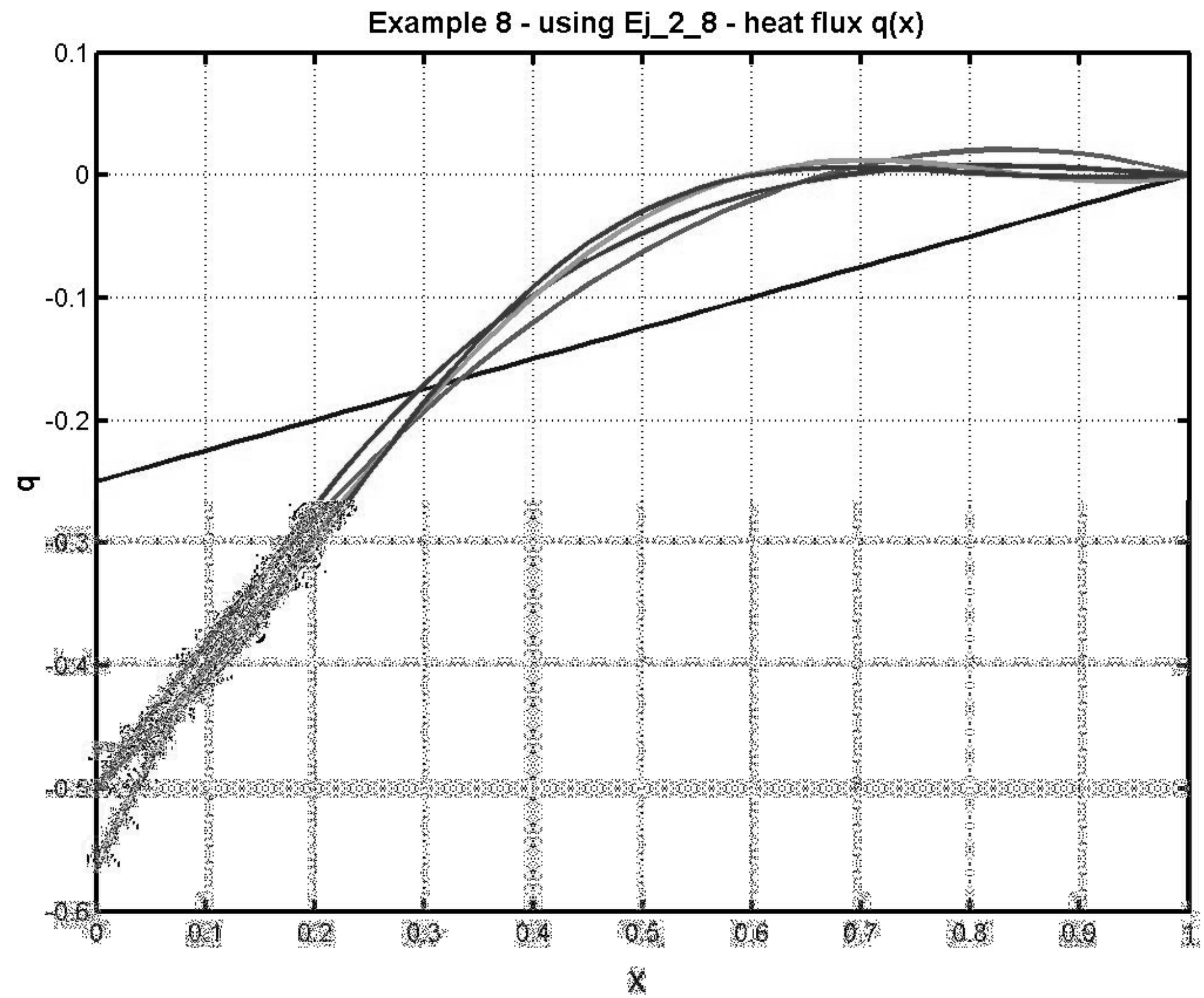
$$f = 0 \quad @\, x = 0 \quad ; \quad q = -k\,\frac{df}{dx} = 0 \quad @\, x = 1$$



Example 8 : SoDE – 2nd order to 1st order
Results – Temperature convergence with M



Example 8 : SoDE – 2nd order to 1st order
Results – heat flux convergence with M



Example 9 : SoDE – Linear elasticity in 2D

Plane stress in

2D elasticity

$$e = \begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} = \mathsf{L}\, \underline{f} \qquad ; \qquad \mathsf{L} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

$$s = \begin{bmatrix} s_x \\ s_y \\ s_{xy} \end{bmatrix} = \frac{E}{1-n^2} \begin{bmatrix} 1 & n & 0 \\ n & 1 & 0 \\ 0 & 0 & \dfrac{1-n}{2} \end{bmatrix} e = \mathsf{D}\, e$$

$$\underline{A}(f) = \begin{bmatrix} \dfrac{\partial s_x}{\partial x} + \dfrac{\partial s_{xy}}{\partial y} + X \\ \dfrac{\partial s_{xy}}{\partial x} + \dfrac{\partial s_y}{\partial y} + Y \end{bmatrix} = \mathsf{L}^{\mathsf{T}} \mathsf{D}\, \mathsf{L}\, \underline{f} + X = 0 \qquad ; \qquad \underline{X}^T = (X, Y)$$

$$\underline{B}(f) = \begin{bmatrix} s_x n_x + s_{xy} n_y - \hat{t}_x \\ s_{xy} n_x + s_y n_y - \hat{t}_y \end{bmatrix} = 0 \qquad \text{on } \Gamma_s$$

$$\underline{B}(f) = \begin{bmatrix} u - \bar{u} \\ v - \bar{v} \end{bmatrix} = 0 \qquad \text{on } \Gamma_f$$

Example 9 : SoDE – Linear elasticity in 2D

Plane stress in

2D elasticity

$$y_1 = \bar{u} \quad ; \quad y_2 = \bar{v} \quad \text{on } \Gamma_f$$

$$\underline{\hat{f}} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \underline{y} + \sum_m \underline{\underline{N_m}}\, \underline{a}_m \quad ; \underline{y}^T = (y_1, y_2)$$

$$\underline{\underline{W_l}} = \begin{bmatrix} W_{l,1} & \\ & W_{l,2} \end{bmatrix} \quad ; \quad \underline{\underline{\overline{W_l}}} = \begin{bmatrix} \overline{W_{l,1}} & \\ & \overline{W_{l,2}} \end{bmatrix}$$

$$\int_\Omega \left(\frac{\partial s_x}{\partial x} + \frac{\partial s_{xy}}{\partial y} + X \right) W_{l,1}\, d\Omega + \int_{\Gamma_s} \left(s_x n_x + s_{xy} n_y - \hat{t}_x \right) \overline{W_{l,1}}\, d\Gamma = 0$$

$$\int_\Omega \left(\frac{\partial s_{xy}}{\partial x} + \frac{\partial s_y}{\partial y} + Y \right) W_{l,2}\, d\Omega + \int_{\Gamma_s} \left(s_{xy} n_x + s_y n_y - \hat{t}_y \right) \overline{W_{l,2}}\, d\Gamma = 0$$

$$\underline{s} = \mathsf{D}\,\mathsf{L}\, \underline{f}$$

Example 9 : SoDE – Linear elasticity in 2D

Gauss - Green or integration by parts

$$-\int_{\Omega}\left(\hat{s}_{x}\frac{\partial W_{l,1}}{\partial x}+\hat{s}_{xy}\frac{\partial W_{l,1}}{\partial y}-W_{l,1}X\right)d\Omega+\int_{\Gamma_s\cup\Gamma_f}\left(\hat{s}_{x}n_x+\hat{s}_{xy}n_y\right)W_{l,1}\,d\Gamma$$

$$+\int_{\Gamma_s}\left(\hat{s}_{x}n_x+\hat{s}_{xy}n_y-\hat{t}_x\right)\overline{W_{l,1}}\,d\Gamma=0$$

$$-\int_{\Omega}\left(\hat{s}_{xy}\frac{\partial W_{l,2}}{\partial x}+\hat{s}_{y}\frac{\partial W_{l,2}}{\partial y}-W_{l,2}Y\right)d\Omega+\int_{\Gamma_s\cup\Gamma_f}\left(\hat{s}_{xy}n_x+\hat{s}_{y}n_y\right)W_{l,2}\,d\Gamma$$

$$+\int_{\Gamma_s}\left(\hat{s}_{xy}n_x+\hat{s}_{y}n_y-\hat{t}_y\right)\overline{W_{l,2}}\,d\Gamma=0$$

Example 9 : SoDE – Linear elasticity in 2D

Choosing the weighted functions as

$$W_{l,1} = W_{l,2} = 0 \qquad \text{on } \Gamma_f$$

$$\overline{W_{l,1}} = -W_{l,1}\Big|_{\Gamma_s}$$

$$\overline{W_{l,2}} = -W_{l,2}\Big|_{\Gamma_s}$$

$$-\int_{\Omega}\left(\hat{s}_x \frac{\partial W_{l,1}}{\partial x} + \hat{s}_{xy} \frac{\partial W_{l,1}}{\partial y} - W_{l,1} X \right) d\Omega + \int_{\Gamma_s} \hat{t}_x W_{l,1} \, d\Gamma = 0$$

$$-\int_{\Omega}\left(\hat{s}_{xy} \frac{\partial W_{l,2}}{\partial x} + \hat{s}_y \frac{\partial W_{l,2}}{\partial y} - W_{l,2} Y \right) d\Omega + \int_{\Gamma_s} \hat{t}_y W_{l,2} \, d\Gamma = 0$$

Example 9 : SoDE – Linear elasticity in 2D

Writing in a compact way

$$\mathrm{L}\,\underline{\hat{f}} = \mathrm{L}\,\underline{y} + \sum_m \mathrm{L}\,\underline{\underline{N_m}}\;\underline{a_m}$$

$$\int_\Omega \left(\mathrm{L}\,\underline{\underline{W_l}}\right)^T \underline{\hat{s}}\, d\Omega - \int_\Omega \underline{\underline{W_l}}\;\underline{X}\; d\Omega - \int_{\Gamma s} \underline{\underline{W_l}}\;\underline{\hat{t}}\, d\Gamma = 0$$

$$\underline{\hat{t}} = \left(\hat{t}_x, \hat{t}_y\right)$$

expressing the stress in terms of displacements

$$\int_\Omega \left(\mathrm{L}\,\underline{\underline{W_l}}\right)^T \mathrm{D}\;\mathrm{L}\,\underline{\hat{f}}\, d\Omega = \int_\Omega \underline{\underline{W_l}}\;\underline{X}\; d\Omega + \underbrace{\int_{\Gamma s} \underline{\underline{W_l}}\;\underline{\hat{t}}\; d\Gamma}_{\text{natural boundary condition}}$$

imposing $\underline{\hat{t}}$ is a natural boundary condition.

Example 9 : SoDE – Linear elasticity in 2D

using the approximation

$$\mathbf{L}\,\underline{\hat{f}} = \mathbf{L}\,\underline{y} + \sum_m \mathbf{L}\,\underline{\underline{N}}_m\,\underline{a}_m$$

$$\int_\Omega \left(\mathbf{L}\,\underline{\underline{W}}_l\right)^T \mathbf{D} \left(\mathbf{L}\,\underline{y} + \sum_m \mathbf{L}\,\underline{\underline{N}}_m\,\underline{a}_m\right) d\Omega = \int_\Omega \underline{\underline{W}}_l\,\underline{X}\;d\Omega + \int_{\Gamma_s} \underline{\underline{W}}_l\,\underline{\hat{t}}\;d\Gamma$$

$$\sum_m \left(\int_\Omega \left(\mathbf{L}\,\underline{\underline{W}}_l\right)^T \mathbf{D}\;\mathbf{L}\,\underline{\underline{N}}_m\;d\Omega \right)\underline{a}_m = \int_\Omega \underline{\underline{W}}_l\,\underline{X}\;d\Omega + \int_{\Gamma_s} \underline{\underline{W}}_l\,\underline{\hat{t}}\;d\Gamma - \int_\Omega \left(\mathbf{L}\,\underline{\underline{W}}_l\right)^T \mathbf{D}\;\mathbf{L}\,\underline{y}\;d\Omega$$

Example 9 : SoDE – Linear elasticity in 2D

linear system

$$\underline{\underline{K}}\,\underline{a} = \underline{f}$$

$$K_{lm} = \int_{\Omega} \left(\mathsf{L}\,\underline{\underline{W}}_l \right)^T \mathsf{D}\ \mathsf{L}\,\underline{\underline{N}}_m \, d\Omega$$

$$f_l = \int_{\Omega} \underline{\underline{W}}_l\,\underline{X}\ d\Omega + \int_{\Gamma_s} \underline{\underline{W}}_l\,\underline{\hat{t}}\ d\Gamma - \int_{\Omega} \left(\mathsf{L}\,\underline{\underline{W}}_l \right)^T \mathsf{D}\ \mathsf{L}\,\underline{y}\ d\Omega$$

Example 9 : SoDE – Linear elasticity in 2D

Remark :

$$\int_{\Omega} \left(\mathsf{L}\, \underline{\underline{W_l}} \right)^T \underline{\hat{s}}\; d\Omega - \int_{\Omega} \underline{\underline{W_l}}\; \underline{X}\; d\Omega - \int_{\Gamma_s} \underline{\underline{W_l}}\; \underline{\hat{t}}\; d\Gamma = 0$$

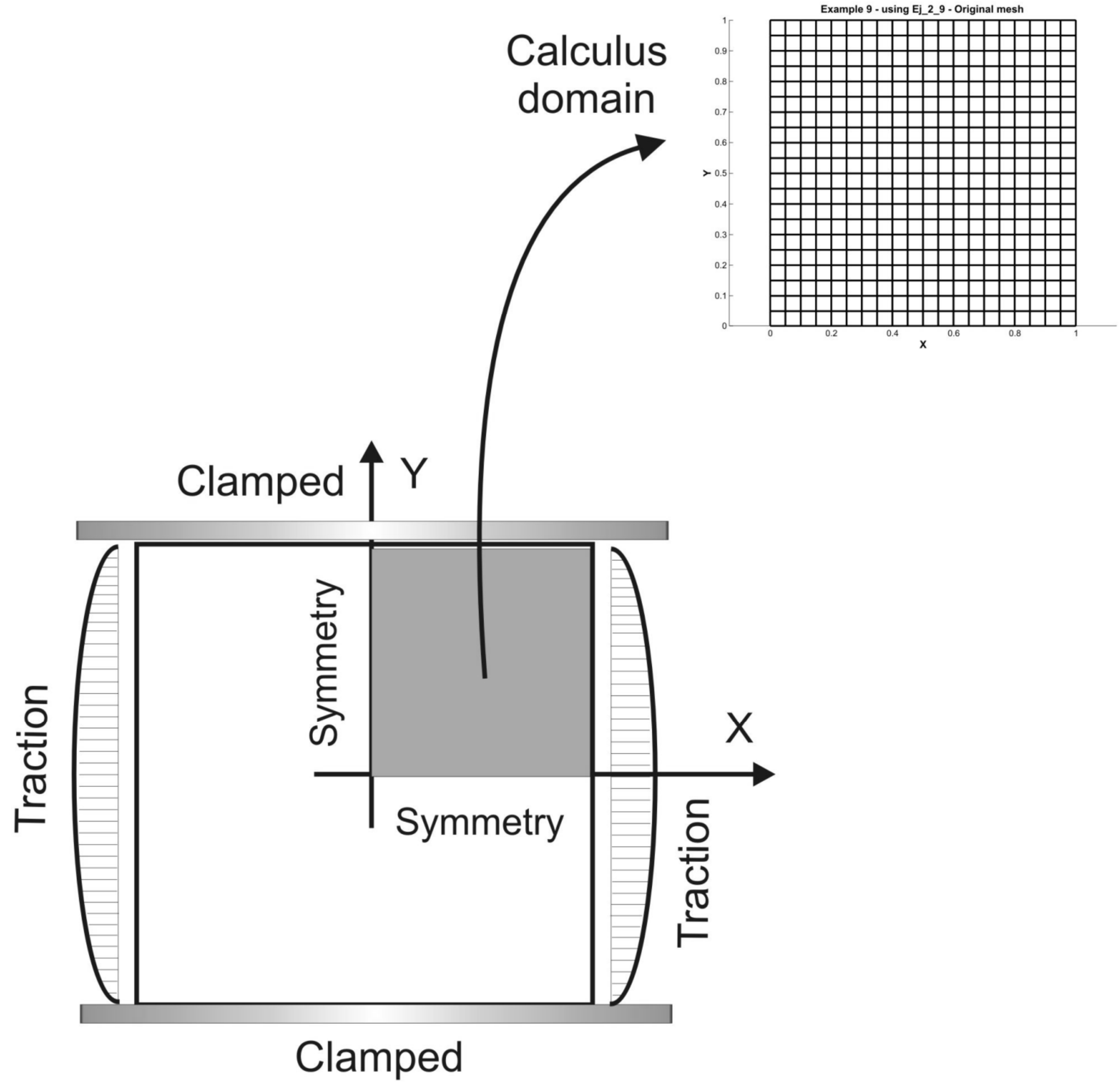
is equivalent to the principle of virtual work

$$\int_{\Omega} \left(\underline{e}^* \right)^T \underline{s}\; d\Omega = \int_{\Omega} \underline{f}^{*T}\, \underline{X}\; d\Omega + \int_{\Gamma_s} \underline{f}^{*T}\, \underline{t}\; d\Gamma$$

using as virtual displacement $\underline{f}^*$ the following definition

$$\underline{f}^* = \underline{\underline{W_l}}\; \underline{d_1}^* \quad \text{with } \underline{d_1}^* \text{ arbitrary and } \quad \underline{e}^* = \mathsf{L}\, \underline{f}^*$$

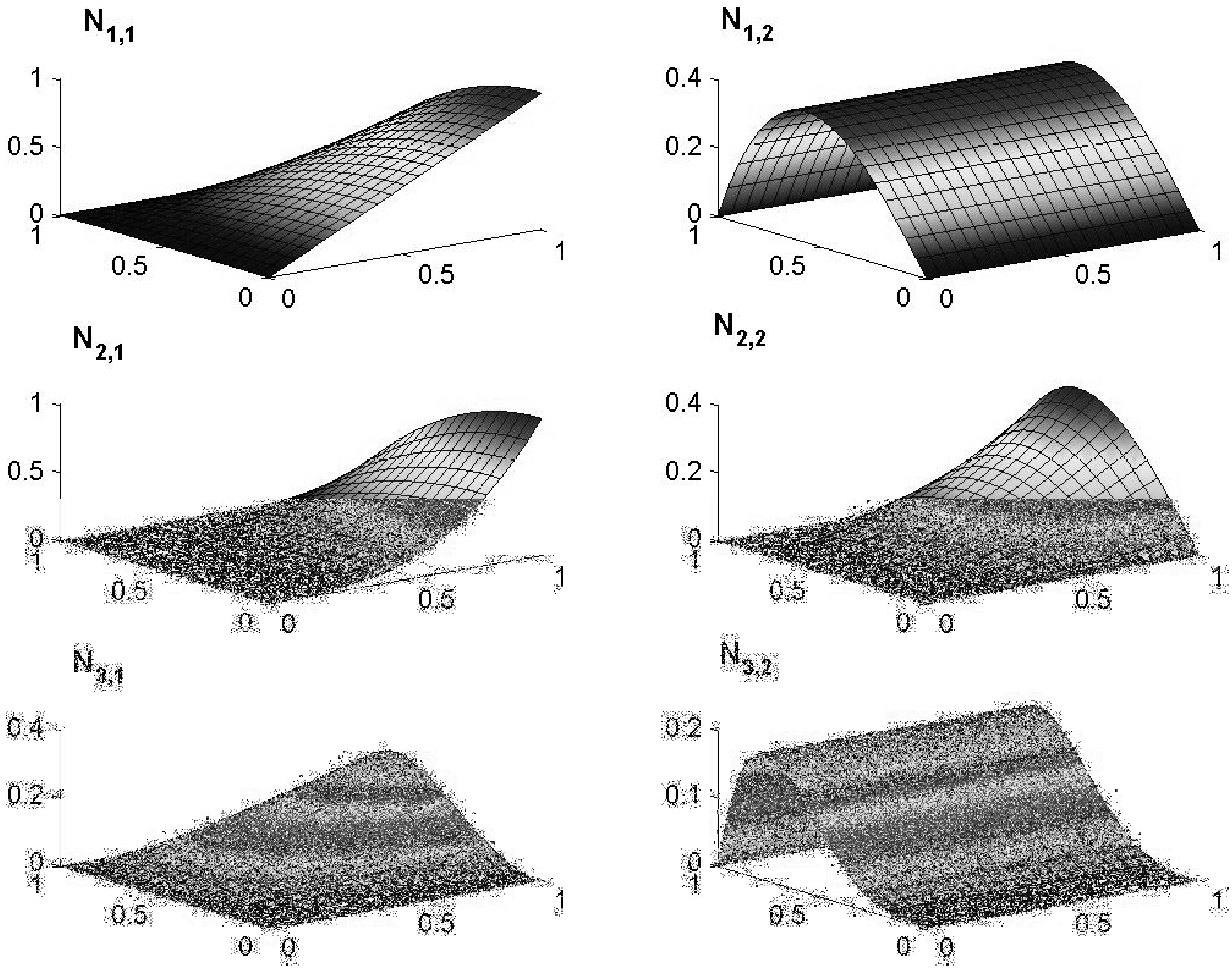
Example 9 : Linear elasticity Problem definition (see Ej_2_9.m routine)



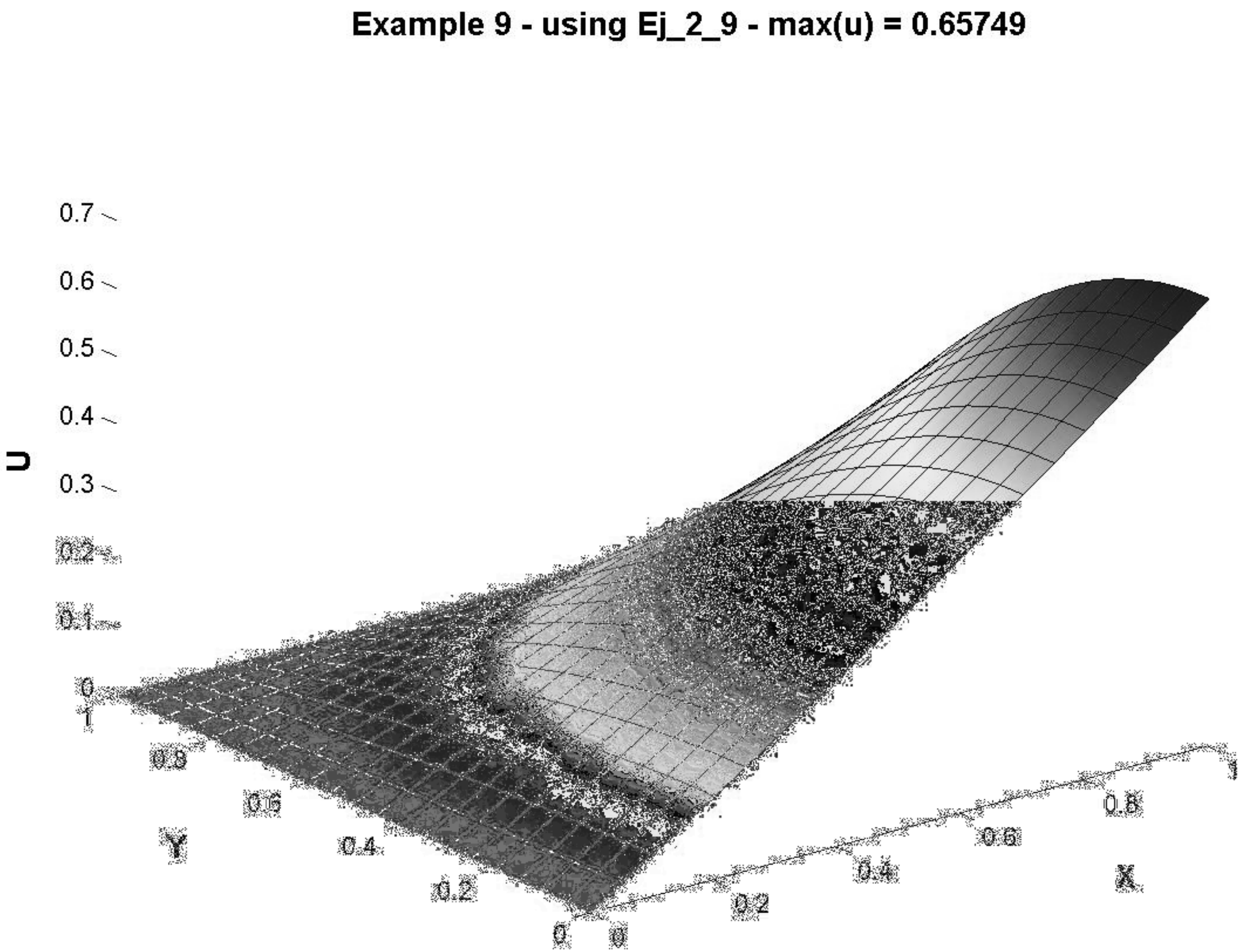
Example 9 : trial functions

```
function [N1,N2,Lx_N1,Ly_N1,Lx_N2,Ly_N2] = shape_function(x,y,mm)
Nxs =strvcat('(x)','(x.^3)','(x.*y.^2)');
dNxsdx =strvcat('1','(3*x.^2)','(y.^2)');
dNxsdy =strvcat('0','0','(2*y.*x)');

Nys =strvcat('(y)','(y.*x.^2)','(y.^3)');
dNysdx =strvcat('0','(2*y.*x)','0');
dNysdy =strvcat('1','(x.^2)','(3*y.^2)');

Nx = Nxs(mm,:);
dNxdx = dNxsdx(mm,:);
dNxdy = dNxsdy(mm,:);

Ny = Nys(mm,:);
dNydx = dNysdx(mm,:);
dNydy = dNysdy(mm,:);

eval(['N1 = (1-y.^2).*' Nx ';']);
eval(['N2 = (1-y.^2).*' Ny ';']);
eval(['Lx_N1 = (1-y.^2).*' dNxdx ';']);
eval(['Ly_N1 = (1-y.^2).*' dNxdy '-' Nx '.*(2*y)' ';']);
eval(['Lx_N2 = (1-y.^2).*' dNydx ';']);
eval(['Ly_N2 = (1-y.^2).*' dNydy '-' Ny '.*(2*y)' ';']);
```

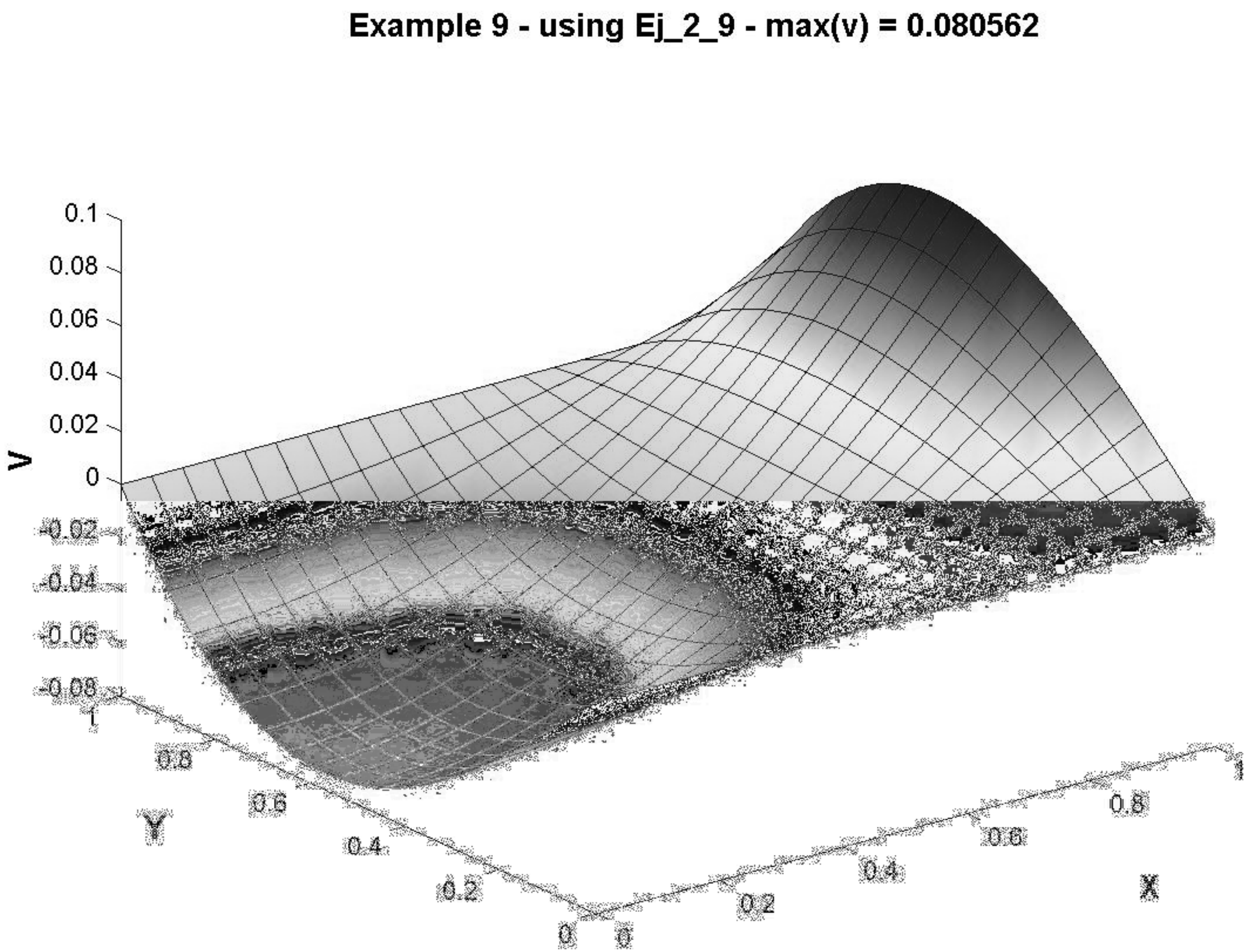
Example 9 : trial functions



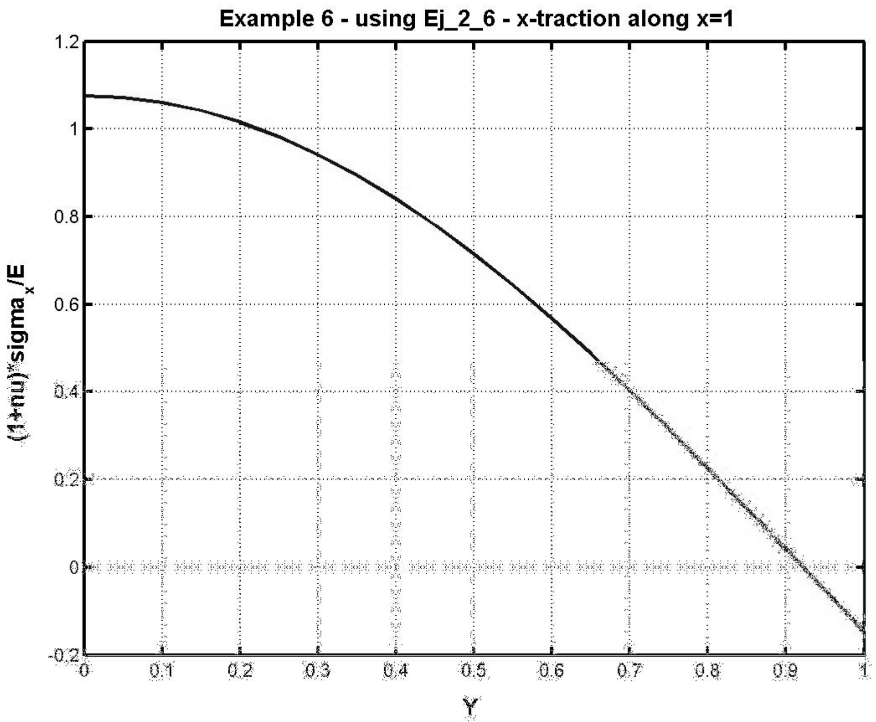
Example 9 : Linear elasticity – Results x-displacement

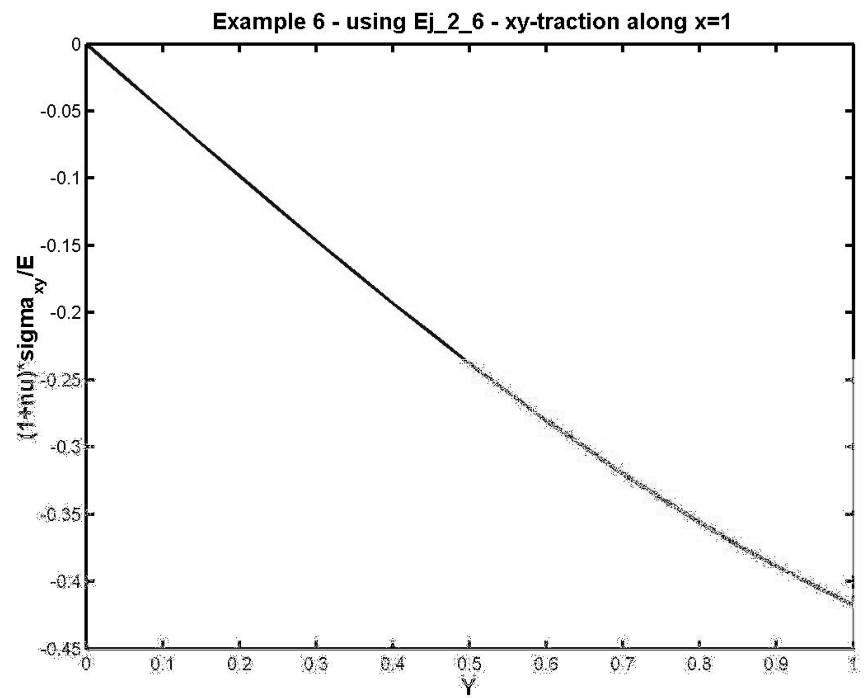


Example 9 : Linear elasticity – Results
y-displacement

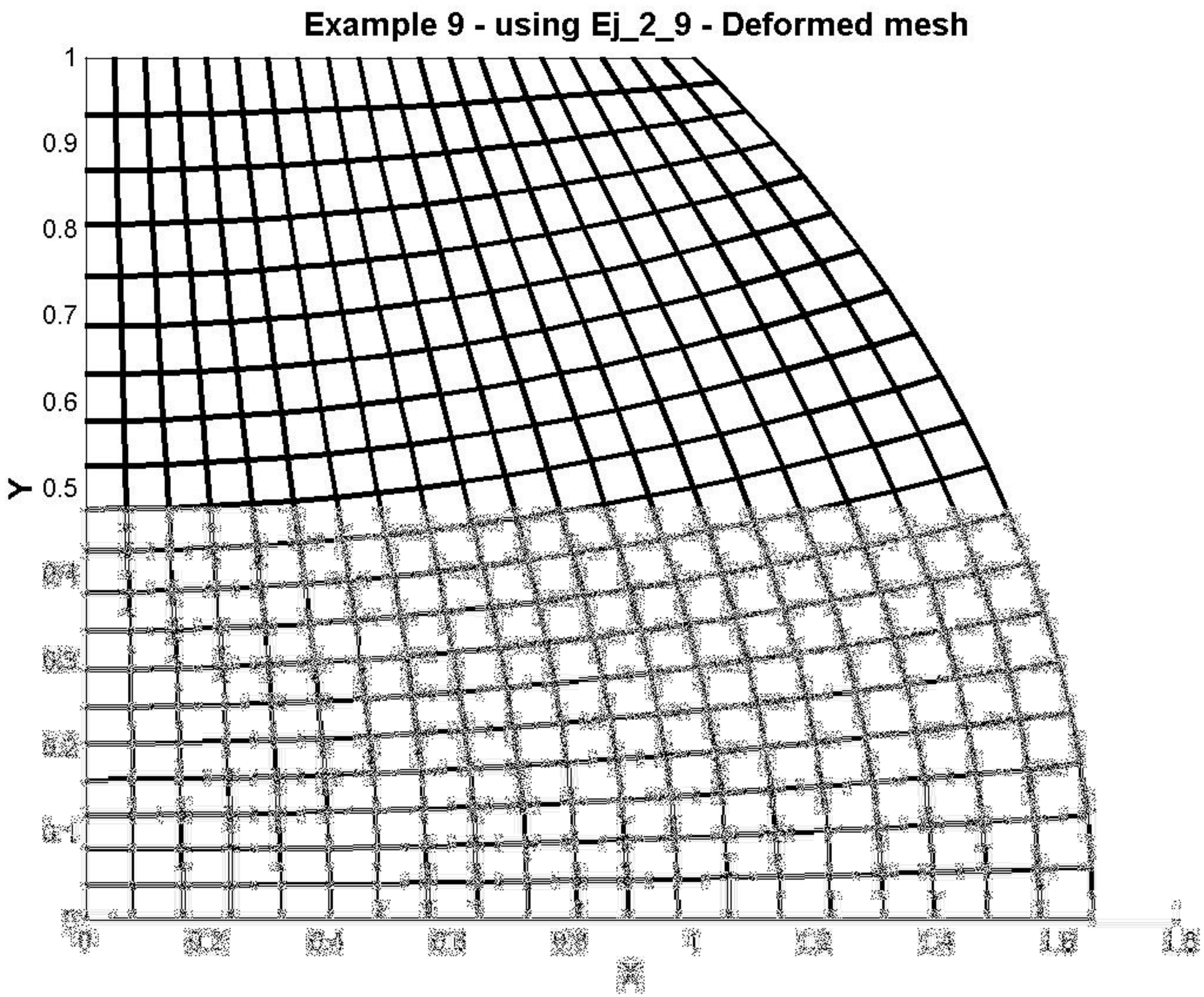


Example 9 : Linear elasticity – Results
Traction at boundary (x=1)





Example 9 : Linear elasticity – Results
Deformed mesh



Non linear problems

$$\text{Non linear problems} \Rightarrow \quad \mathrm{K(a)\ a = f}$$

$$\text{Example : Find } \hat{f}(x) \text{ solution of the following PDE}$$

$$\frac{\partial}{\partial \mathrm{x}}\left(k(f)\frac{\partial f}{\partial \mathrm{x}}\right)+\frac{\partial}{\partial \mathrm{y}}\left(k(f)\frac{\partial f}{\partial \mathrm{y}}\right)+Q=0 \qquad \text{in } \Omega : \left\{\mathrm{x};\, x \in \Re^2\right\}$$

$$f=\bar{f} \quad \text{on } \Gamma_f \qquad ; \qquad k\frac{\partial f}{\partial h}=-\bar{q} \quad \text{on } \Gamma_q$$

$$\hat{f}=y+\sum_m a_m N_m$$

$$y=\bar{f} \quad \text{on } \Gamma_f \quad ; \quad N_m=0 \quad \text{on } \Gamma_f$$

$$\int_\Omega W_l\left(\frac{\partial}{\partial \mathrm{x}}\left(k(\hat{f})\frac{\partial \hat{f}}{\partial \mathrm{x}}\right)+\frac{\partial}{\partial \mathrm{y}}\left(k(\hat{f})\frac{\partial \hat{f}}{\partial \mathrm{y}}\right)+Q\right)d\Omega+\int_{\Gamma_q}\overline{W_l}\left(k(\hat{f})\frac{\partial \hat{f}}{\partial h}+\bar{q}\right)d\Gamma=0$$

Non linear problems

$$\int_{\Omega} W_l \left(\frac{\partial}{\partial \mathrm{x}} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial \mathrm{x}} \right) + \frac{\partial}{\partial \mathrm{y}} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial \mathrm{y}} \right) + Q \right) d\Omega + \int_{\Gamma_q} \overline{W_l} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial h} + \bar{q} \right) d\Gamma = 0$$

integration by parts (Gauss - Green)

$$\int_{\Omega} \left(\frac{\partial W_l}{\partial \mathrm{x}} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial \mathrm{x}} \right) + \frac{\partial W_l}{\partial \mathrm{y}} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial \mathrm{y}} \right) - W_l Q \right) d\Omega -$$

$$- \int_{\Gamma_q \cup \Gamma_f} W_l \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial h} \right) d\Gamma - \int_{\Gamma_q} \overline{W_l} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial h} + \bar{q} \right) d\Gamma = 0$$

$$\overline{W_l} = -W_l \Big|_{\Gamma_q} \qquad ; \qquad W_l \Big|_{\Gamma_f} = 0$$

$$\int_{\Omega} \left(\frac{\partial W_l}{\partial \mathrm{x}} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial \mathrm{x}} \right) + \frac{\partial W_l}{\partial \mathrm{y}} \left(k\left(\hat{f}\right) \frac{\partial \hat{f}}{\partial \mathrm{y}} \right) - W_l Q \right) d\Omega + \int_{\Gamma_q} W_l \bar{q} \; d\Gamma = 0$$

Non linear problems

$$K_{lm}\left(\underline{a}^0\right)=\int_\Omega\left(\frac{\partial W_l}{\partial x}\left(k\left(\hat{f}^0\right)\frac{\partial N_m}{\partial x}\right)+\frac{\partial W_l}{\partial y}\left(k\left(\hat{f}^0\right)\frac{\partial N_m}{\partial y}\right)\right)d\Omega$$

$$f_l\left(\underline{a}^0\right)=\int_\Omega W_l Q\, d\Omega-\int_{\Gamma_q} W_l \bar{q}\; d\Gamma-$$

$$-\int_\Omega\left(\frac{\partial W_l}{\partial x}\left(k\left(\hat{f}^0\right)\frac{\partial y}{\partial x}\right)+\frac{\partial W_l}{\partial y}\left(k\left(\hat{f}^0\right)\frac{\partial y}{\partial y}\right)\right)d\Omega+$$

In general, the iterations follow as:

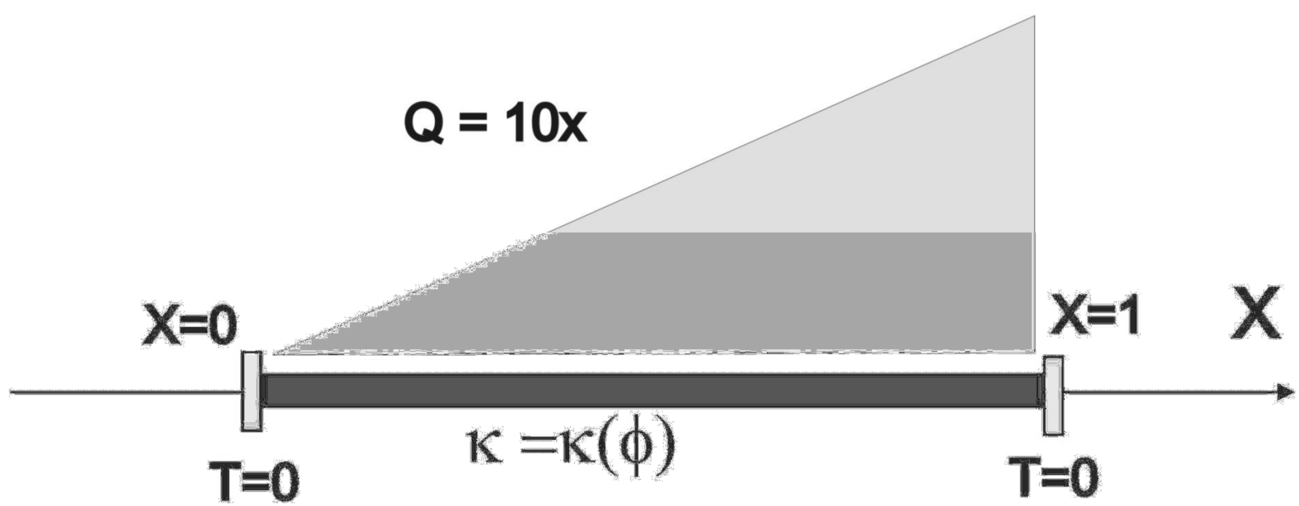
$$K(a^{n-1})a^n = f^{n-1}$$

Example 10 : Non linear heat equation

$$\frac{d}{dx}\left(k\,\frac{df}{dx}\right)+Q=0$$

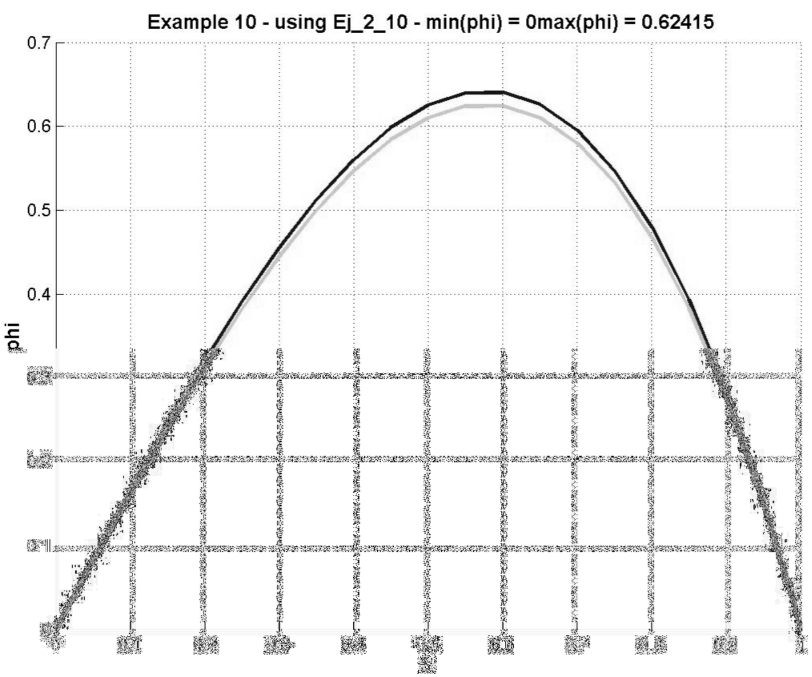
$$\Omega:\{x;0\le x\le 1\}\quad ;\quad k=1+0.1f\quad ;\quad Q=10x$$

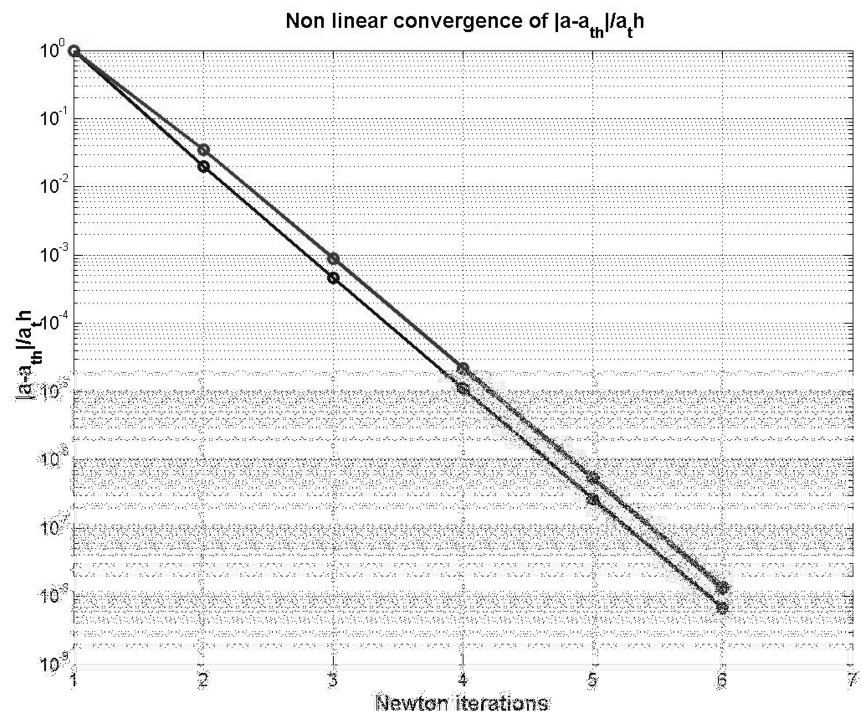
$$f=0\quad @\,x=0\quad ;\quad f=0\quad @\,x=1$$



Example 10 : Non linear convergence of solution

M=2 – convergence of coefficients **a** with Galerkin using Ej_2_10.m routine





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