Approximation to PDE
by narrow-base trial functions

- Defining the PDE problem
- Introduction to the PDE approximation
- Weak formulation and the Galerkin method
- Some 1D problems.
- Generalization to 2D and 3D problems
- 2D heat equation
- 2D elastic stress analysis
- Finite differences as a special case of finite elements

Defining the PDE problem

$$A(f) = 0 \qquad \text{in } \Omega$$

L : linear differenti al operator

$$p \neq p(f)$$

$$\mathsf{L}(f) = \frac{\partial}{\partial x}\left(k\,\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial f}{\partial y}\right)$$

$$p = Q$$

$$k \neq k(f)$$

$$B(f) = \mathsf{M}(f) + \mathrm{r} = 0 \qquad \text{in } \Gamma$$

$$\begin{cases} \mathsf{M}(f) = f & ; \quad r = -\overline{f} \qquad \text{on } \Gamma_f \quad \text{DIRICHLET} \\ \mathsf{M}(f) = -k\,\dfrac{\partial f}{\partial h} & ; \quad r = -\overline{q} \qquad \text{on } \Gamma_q \quad \text{NEUMANN} \end{cases}$$

Introduction to the PDE approximation
How to choose the trial functions

$$f(x) \cong \hat{f}(x) = \sum_{m} f_m N_m(x)$$

$$\text{on } \Gamma \quad \begin{cases} \mathrm{M}(y) = -r \\ \mathrm{M}(N_m) = 0 \quad ; \quad m = 1, 2, \cdots \end{cases}$$

$$\hat{f}(x) = f(x) \quad \text{for all } x \in \Gamma$$

Introduction to the PDE approximation
The trial functions and its derivatives

$$f(x) \cong \hat{f}(x) = \sum_m f_m N_m(x)$$

$$\mathsf{L}\hat{f} = \sum_m f_m \, \mathsf{L} N_m$$

$$\frac{\partial f}{\partial x} \cong \frac{\partial \hat{f}}{\partial x} = \sum_m f_m \frac{\partial N_m}{\partial x}$$

$$\frac{\partial^2 f}{\partial x} \cong \frac{\partial^2 \hat{f}}{\partial x} = \sum_m f_m \frac{\partial^2 N_m}{\partial x}$$

N_m needs the regularity imposed by L

to relax it we redefine the problem in a weak form

Approximation by weighted residual
Weak formulation and the Galerkin method

$$R_\Omega = A(f) = \mathsf{L}\hat{f} + p = \sum_m f_m \,\mathsf{L}\, N_m + p$$

$$R_\Gamma = B(f) = \mathsf{M}(f) + r = \sum_m f_m \,\mathsf{M}\, N_m + r$$

$$\int_\Omega W_l \; R_\Omega \; d\Omega + \int_\Gamma \overline{W}_l \; R_\Gamma \; d\Gamma = 0$$

$$\int_\Omega W_l \left(\sum_m f_m \,\mathsf{L}\, N_m + p \right) d\Omega + \int_\Gamma \overline{W}_l \left(\sum_m f_m \,\mathsf{M}\, N_m + r \right) d\Gamma = 0$$

$$\int_\Omega W_l \left(\sum_m f_m \,\mathsf{L}\, N_m \right) d\Omega + \int_\Gamma \overline{W}_l \left(\sum_m f_m \,\mathsf{M}\, N_m \right) d\Gamma = -\int_\Omega W_l \, p \; d\Omega - \int_\Gamma \overline{W}_l \; r \; d\Gamma$$

$$K_{lm} f_m = f_l$$

$$K_{lm} = \int_\Omega W_l \,\mathsf{L}\, N_m \; d\Omega + \int_\Gamma \overline{W}_l \,\mathsf{M}\, N_m \; d\Gamma$$

$$f_l = -\int_\Omega W_l \; p \; d\Omega - \int_\Gamma \overline{W}_l \; r \; d\Gamma \qquad l, m = 1, 2, \cdots, M$$

Weak formulation and functional regularity

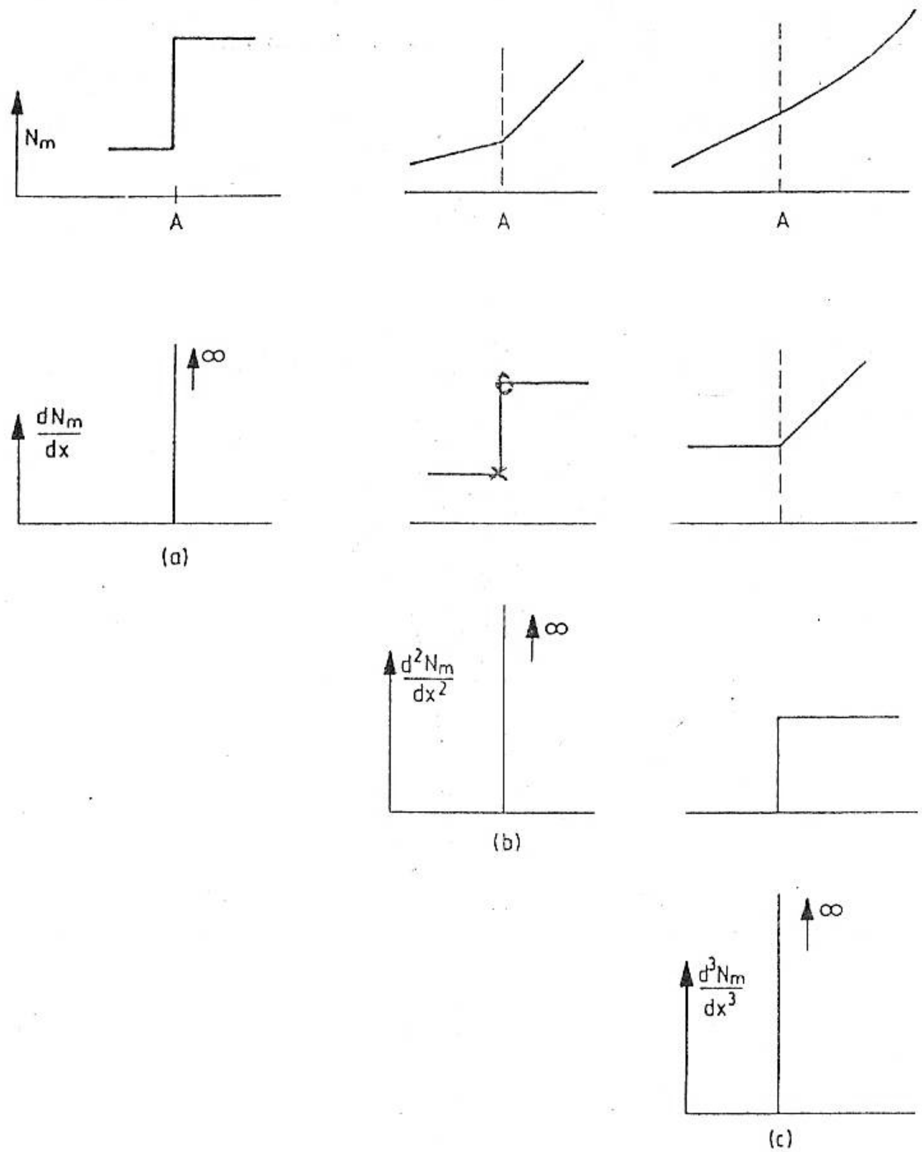


FIGURE 3.4. Behavior of three types of one-dimensional shape functions and their derivatives near the junction A of two elements.

Weak formulation and functional regularity

$$\underbrace{\int_{\Omega} W_l \left(\mathsf{L}\hat{f} + \mathrm{p}\right) d\Omega}_{\text{strong form}} = \underbrace{\overbrace{\int_{\Omega} \mathsf{C} W_l \, \mathsf{D}\hat{f} \; d\Omega + \int_{\Gamma} W_l \, \mathsf{E}\hat{f} \; d\Gamma}^{\text{integration by parts}} + \int_{\Omega} W_l \;\, p d\Omega = 0}_{\text{weak form}}$$

$$\int_{\Omega} W_l \left(\mathsf{L}\hat{f} + \mathrm{p}\right) d\Omega + \int_{\Gamma} \overline{W_l} \left(\mathsf{M}\hat{f} + \mathrm{r}\right) d\Gamma =$$

$$= \int_{\Omega} \mathsf{C} W_l \, \mathsf{D}\hat{f} \; d\Omega + \int_{\Omega} W_l \;\, p d\Omega + \int_{\Gamma} W_l \, \mathsf{E}\hat{f} \; d\Gamma + \int_{\Gamma} \overline{W_l} \left(\mathsf{M}\hat{f} + \mathrm{r}\right) d\Gamma = 0$$

In this way it is feasible to reduce the regularity requirements for the functional spaces balancing the cost of the computations between weight and trial functions

Some 1D problems
Again solving example 5 now with FEM

Find $\hat{f}(x)$ solution of the following ODE

$$\frac{\mathrm{d}^2 f}{\mathrm{dx}^2} - f = 0 \qquad \text{in } \Omega : \{\mathrm{x}; 0 \le \mathrm{x} \le 1\}$$

$$f(x=0) = 0 \quad ; \quad f(x=1) = 1 \quad \text{in } \Gamma : \{\mathrm{x} = 0, \mathrm{x} = 1\}$$

$$f(x) \cong \hat{f}(x) = \sum_m f_m N_m(x) \text{ using for } N_m \text{ piecewise linear}$$

$$\int_0^1 W_l \left(\frac{\mathrm{d}^2 f}{\mathrm{dx}^2} - f \right) dx + \left[\overline{W}_l R_\Gamma \right]_{x=0} + \left[\overline{W}_l R_\Gamma \right]_{x=1} = 0 \quad \text{for } l = 1,2,\ldots M+1$$

$$-\int_0^1 \frac{dW_l}{dx} \frac{d\hat{f}}{dx} dx - \int_0^1 W_l \; \hat{f} \; dx + \left[W \frac{d\hat{f}}{dx} \right]_{x=1} - \left[W \frac{d\hat{f}}{dx} \right]_{x=0} +$$

$$+ \left[\overline{W}_l \left(\hat{f} - 0 \right) \right]_{x=0} + \left[\overline{W}_l \left(\hat{f} - 1 \right) \right]_{x=1} = 0$$

Example 5 by piecewise linear Galerkin

Choosing $\overline{W}_l\Big|_{\Gamma} = -W_l\Big|_{\Gamma}$ & $W_l\Big|_{\Gamma_f} = 0$

with $\Gamma = \Gamma_f \cup \Gamma_q$

$$-\int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx - \int_0^1 W_l\,\hat{f}\,dx + \left[W_l\,\frac{d\hat{f}}{dx}\right]_{x=1} - \left[W_l\,\frac{d\hat{f}}{dx}\right]_{x=0} -$$

$$-\underbrace{\left[W_l\hat{f}\right]_{x=0}}_{=0\,(\text{DIRICHLET})} - \underbrace{\left[W_l\left(\hat{f}-1\right)\right]_{x=1}}_{=0\,(\text{DIRICHLET})} = 0$$

$$\Rightarrow \qquad \int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx + \int_0^1 W_l\,\hat{f}\,dx = \left[W_l\,\frac{d\hat{f}}{dx}\right]_{x=0}^{x=1}$$

the rhs term may be cancelled out because $W_l\Big|_{\Gamma_f} = 0$

However it is possible to keep it including $\left.\frac{d\hat{f}}{dx}\right|_{\mathrm{x}=0}^{\mathrm{x}=1}$ as unknowns

Example 5 by piecewise linear Galerkin

$$\int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx + \int_0^1 W_l\ \hat{f}\ dx = \left[W_l\ \frac{d\hat{f}}{dx}\right]_{x=0}^{x=1}$$

$$\Rightarrow \quad \underline{\underline{K}}\ \underline{f}\ = \underline{f}$$

$$K_{lm} = \int_0^1 \frac{dW_l}{dx}\frac{dN_m}{dx}dx + \int_0^1 W_l\ N_m dx$$

$$f_l \quad = \left[W_l\ \frac{d\hat{f}}{dx}\right]_{x=0}^{x=1} \qquad ; \quad l, m = 1, 2, \ldots M, M+1$$

$$\underline{f}\ = \{f_1, f_2, \cdots, f_M, f_{M+1}\}$$

Example 5 by piecewise linear Galerkin

$$N_i^e = \frac{h^e - c}{h^e} \quad ; \quad N_j^e = \frac{c}{h^e} \quad ; \quad c = x - x_i \quad ; \quad h^e = x_j - x_i$$

$$K_{lm} = \sum_{e=1}^{E} K_{lm}^e \quad ; \quad K_{lm}^e = \begin{cases} 0 \quad \text{if } l,m \notin \text{element } e \\ K_{ij}^e = K_{ji}^e = \int_0^{h^e} \left(\frac{dN_i^e}{dx}\frac{dN_j^e}{dx} + N_i^e N_j^e \right) dc = \\ \qquad = -\frac{1}{h^e} + \frac{h^e}{6} \\ K_{ii}^e = K_{jj}^e = \int_0^{h^e} \left(\left(\frac{dN_i^e}{dx} \right)^2 + \left(N_i^e \right)^2 \right) dc = \\ \qquad = \frac{1}{h^e} + \frac{h^e}{3} \end{cases}$$

Example 5 by piecewise linear Galerkin

Assembling the system (gathering) $\Rightarrow \underline{\underline{K}}\,\underline{f} = \underline{f}$

$\underline{\underline{K}}$ is gathered as

$$\underbrace{\begin{bmatrix} \frac{1}{h^e}+\frac{h^e}{3} & -\frac{1}{h^e}+\frac{h^e}{6} & 0 & 0 \\ -\frac{1}{h^e}+\frac{h^e}{6} & \frac{1}{h^e}+\frac{h^e}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h^e}+\frac{h^e}{3} & -\frac{1}{h^e}+\frac{h^e}{6} & 0 \\ 0 & -\frac{1}{h^e}+\frac{h^e}{6} & \frac{1}{h^e}+\frac{h^e}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 2}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{h^e}+\frac{h^e}{3} & -\frac{1}{h^e}+\frac{h^e}{6} \\ 0 & 0 & -\frac{1}{h^e}+\frac{h^e}{6} & \frac{1}{h^e}+\frac{h^e}{3} \end{bmatrix}}_{\text{ELEMENT 3}}$$

f is gathered as

$$\underbrace{\begin{bmatrix} \left.\frac{d\hat{f}}{dx}\right|_{x=0} & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{ELEMENT 2}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \left.\frac{d\hat{f}}{dx}\right|_{x=1} \end{bmatrix}}_{\text{ELEMENT 3}}$$

the system may be solved for 2 unknowns $\{f_2, f_3\}$ or for 4 unknowns $\left\{\frac{df_1}{dx}, f_2, f_3, \frac{df_4}{dx}\right\}$

Example 5 by piecewise linear Galerkin

with 4 unknowns

$$\begin{bmatrix} \frac{1}{h^e}+\frac{h^e}{3} & -\frac{1}{h^e}+\frac{h^e}{6} & 0 & 0 \\ -\frac{1}{h^e}+\frac{h^e}{6} & 2\left(\frac{1}{h^e}+\frac{h^e}{3}\right) & -\frac{1}{h^e}+\frac{h^e}{6} & 0 \\ 0 & -\frac{1}{h^e}+\frac{h^e}{6} & 2\left(\frac{1}{h^e}+\frac{h^e}{3}\right) & -\frac{1}{h^e}+\frac{h^e}{6} \\ 0 & 0 & -\frac{1}{h^e}+\frac{h^e}{6} & \frac{1}{h^e}+\frac{h^e}{3} \end{bmatrix} \begin{bmatrix} f_1=0 \\ f_2 \\ f_3 \\ f_4=1 \end{bmatrix} = \begin{bmatrix} -\frac{df_1}{dx} \\ 0 \\ 0 \\ \frac{df_4}{dx} \end{bmatrix}$$

with 2 unknowns

$$\begin{bmatrix} 2\left(\frac{1}{h^e}+\frac{h^e}{3}\right) & -\frac{1}{h^e}+\frac{h^e}{6} \\ -\frac{1}{h^e}+\frac{h^e}{6} & 2\left(\frac{1}{h^e}+\frac{h^e}{3}\right) \end{bmatrix} \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{h^e}+\frac{h^e}{6} \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{df_1}{dx} \\ \frac{df_4}{dx} \end{bmatrix} = \begin{bmatrix} \frac{1}{h^e}+\frac{h^e}{3} & -\frac{1}{h^e}+\frac{h^e}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{h^e}+\frac{h^e}{6} & \frac{1}{h^e}+\frac{h^e}{3} \end{bmatrix} \begin{bmatrix} f_1=0 \\ f_2 \\ f_3 \\ f_4=1 \end{bmatrix}$$

Example 5 by piecewise linear Galerkin (routine Ej_4_1.m)

Assembling

```
Ke = zeros(numel,2,2);
fe = zeros(numel,2);

% loop over each element
for k=1:numel
   psi = (0:10)'/10;
   node1 = icone(k,1);
   node2 = icone(k,2);
   xx  = psi*(xnod(node2,1)-xnod(node1,1))+xnod(node1,1);
   [N,L_N] = shape_function(xx);
   for mi=1:2
      for mj=1:2
         Ke(k,mi,mj) = Ke(k,mi,mj) + trapz(xx,N(:,mi).*N(:,mj));
         Ke(k,mi,mj) = Ke(k,mi,mj) + trapz(xx,L_N(:,mi).*L_N(:,mj));
      end
   end
end
```

Example 5 by piecewise linear Galerkin (routine Ej_4_1.m)

```
% gather Ke and fe in Kg and fg
Kg = zeros(Nx+1,Nx+1);
fg = zeros(Nx+1,1);
for k=1:numel
   for mi=1:nen
      for mj=1:nen
         Kg(icone(k,mi),icone(k,mj))=Kg(icone(k,mi),icone(k,mj)) + Ke(k,mi,mj);
      end
      fg(icone(k,mi),1) = fg(icone(k,mi),1) + fe(k,mi);
   end
end
```

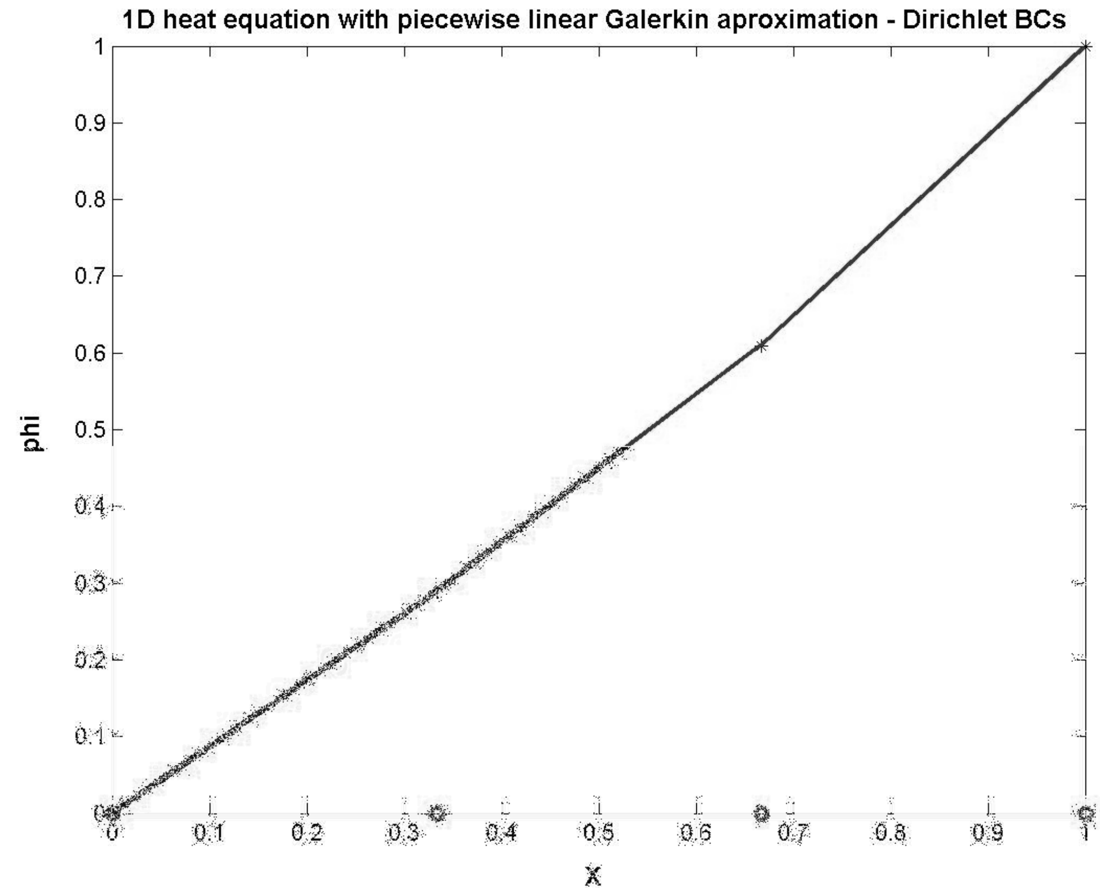
Gathering

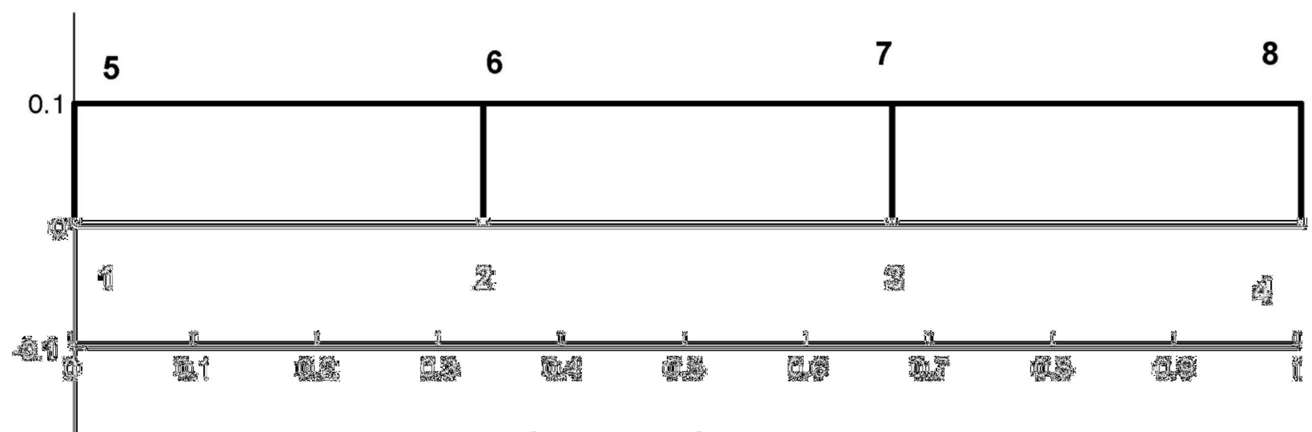
```
% reducing the global system for fixations
free = (2:Nx)';
fg(free,1) = fg(free,1) - Kg(free,1)*phi_lef - Kg(free,Nx+1)*phi_rig;
Kg = Kg(free,free);
fg = fg(free,1);
% solver
phi = Kg\fg;

phi = [phi_lef;phi;phi_rig];
```

Applied BC and solving

Example 5: Heat equation in 1D with FEM Results





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Exact solution to Example 5

In order to analyze the error we solve the ODE for the exact solution.

$$\frac{\mathrm{d}^2 f}{\mathrm{dx}^2} - f = 0 \quad \text{admits solutions of the form } f = A_k e^{l_k x}$$

replacing in the ODE we get

$$\left(l_k^{\,2} - 1\right)\left(A_k e^{l_k x}\right) = 0 \Rightarrow l = \pm 1$$

using $f(x=0)=0$

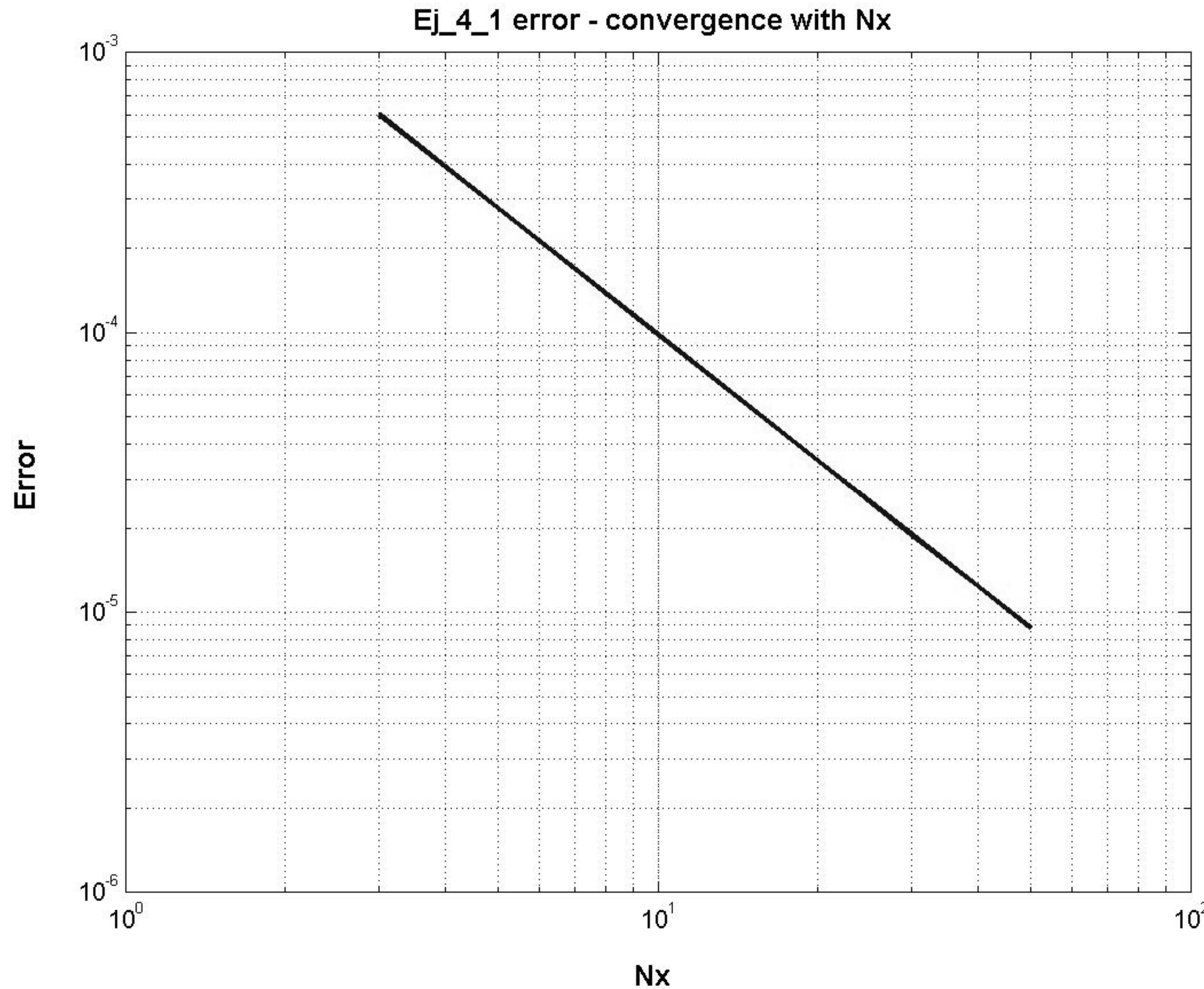
$$f(x=0) = A_1 e^{x} + A_2 e^{-x}\Big|_{x=0} = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$$

using $f(x=1)=1$

$$f(x=1) = A_1 e^{x} + A_2 e^{-x}\Big|_{x=1} = A_1 e + A_2 e^{-1} = 1 \Rightarrow A_1 = \frac{1}{e - \frac{1}{e}}$$

$$\therefore \quad f = \frac{1}{e - \frac{1}{e}}\left(e^{x} - e^{-x}\right) = \frac{e^{x} - e^{-x}}{e - \frac{1}{e}}$$

Example 5: Heat equation in 1D with FEM Results



[phi , phi_th] = […

0	**0**
2.885539967914507e-001	**2.889212154417394e-001**
6.097683214596928e-001	**6.102431352945081e-001**
1.000000000000000e+000	**1.000000000000000e+000]**

Some 1D problems
Example 5 now with Neumann BC at x=1

Find $\hat{f}(x)$ solution of the following ODE

$$\frac{\mathrm{d}^2 f}{\mathrm{dx}^2} - f = 0 \qquad \text{in } \Omega : \{\mathrm{x}; 0 \le \mathrm{x} \le 1\}$$

$$f(x=0) = 0 \quad ; \quad \frac{\mathrm{d}f}{\mathrm{dx}}(x=1) = 1 \quad \text{in } \Gamma : \{\mathrm{x}=0, \mathrm{x}=1\}$$

$$f(x) \cong \hat{f}(x) = \sum_m f_m N_m(x) \text{ using for } N_m \text{ piecewise linear}$$

$$\int_0^1 W_l \left(\frac{\mathrm{d}^2 f}{\mathrm{dx}^2} - f \right) dx + \left[\overline{W}_l R_\Gamma \right]_{x=0} + \left[\overline{W}_l R_\Gamma \right]_{x=1} = 0 \quad \text{for } l = 1,2,\ldots M+1$$

$$-\int_0^1 \frac{dW_l}{dx} \frac{d\hat{f}}{dx} dx - \int_0^1 W_l \, \hat{f} \, dx + \left[W \frac{d\hat{f}}{dx} \right]_{x=1} - \left[W \frac{d\hat{f}}{dx} \right]_{x=0} +$$

$$+ \left[\overline{W}_l \left(\hat{f} - 0 \right) \right]_{x=0} + \left[\overline{W}_l \left(\frac{d\hat{f}}{dx} - 1 \right) \right]_{x=1} = 0$$

Example 5 by piecewise linear Galerkin

Choosing $\overline{W}_l\Big|_{\Gamma} = -W_l\Big|_{\Gamma}$ & $W_l\Big|_{\Gamma_f} = 0$ with $\Gamma = \Gamma_f \cup \Gamma_q$

$$-\int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx - \int_0^1 W_l\ \hat{f}\ dx - W_l\ \frac{d\hat{f}}{dx}\Bigg|_{x=0} - \underbrace{\left[W_l\hat{f}\right]_{x=0}}_{=0\ (\text{DIRICHLET})} + \left[W_l\right]_{x=1} = 0$$

$$\Rightarrow \qquad \int_0^1 \frac{dW_l}{dx}\frac{d\hat{f}}{dx}dx + \int_0^1 W_l\ \hat{f}\ dx + W_l\ \frac{d\hat{f}}{dx}\Bigg|_{x=0} - \left[W_l\right]_{x=1} = 0$$

Again we retain $\left.\dfrac{d\hat{f}}{dx}\right|_{x=0}$ as an unknown

$$K_{lm} = \int_0^1 \frac{dW_l}{dx}\frac{dN_m}{dx}dx + \int_0^1 W_l\ N_m dx$$

$$f_l \quad = \left[W_l\right]_{x=1} - W_l\ \frac{d\hat{f}}{dx}\Bigg|_{x=0} \qquad ; \qquad l, m = 1,2,\ldots M, M+1$$

Exact solution to Example 5 with Neumann BC

$$\frac{\mathrm{d}^2 f}{\mathrm{dx}^2} - f = 0 \qquad \text{admits solutions of the form } f = A_k e^{l_k x}$$

replacing in the ODE we get

$$\left(l_k^{\ 2} - 1\right)\left(A_k e^{l_k x}\right) = 0 \Rightarrow l = \pm 1$$

using $f(x=0) = 0$

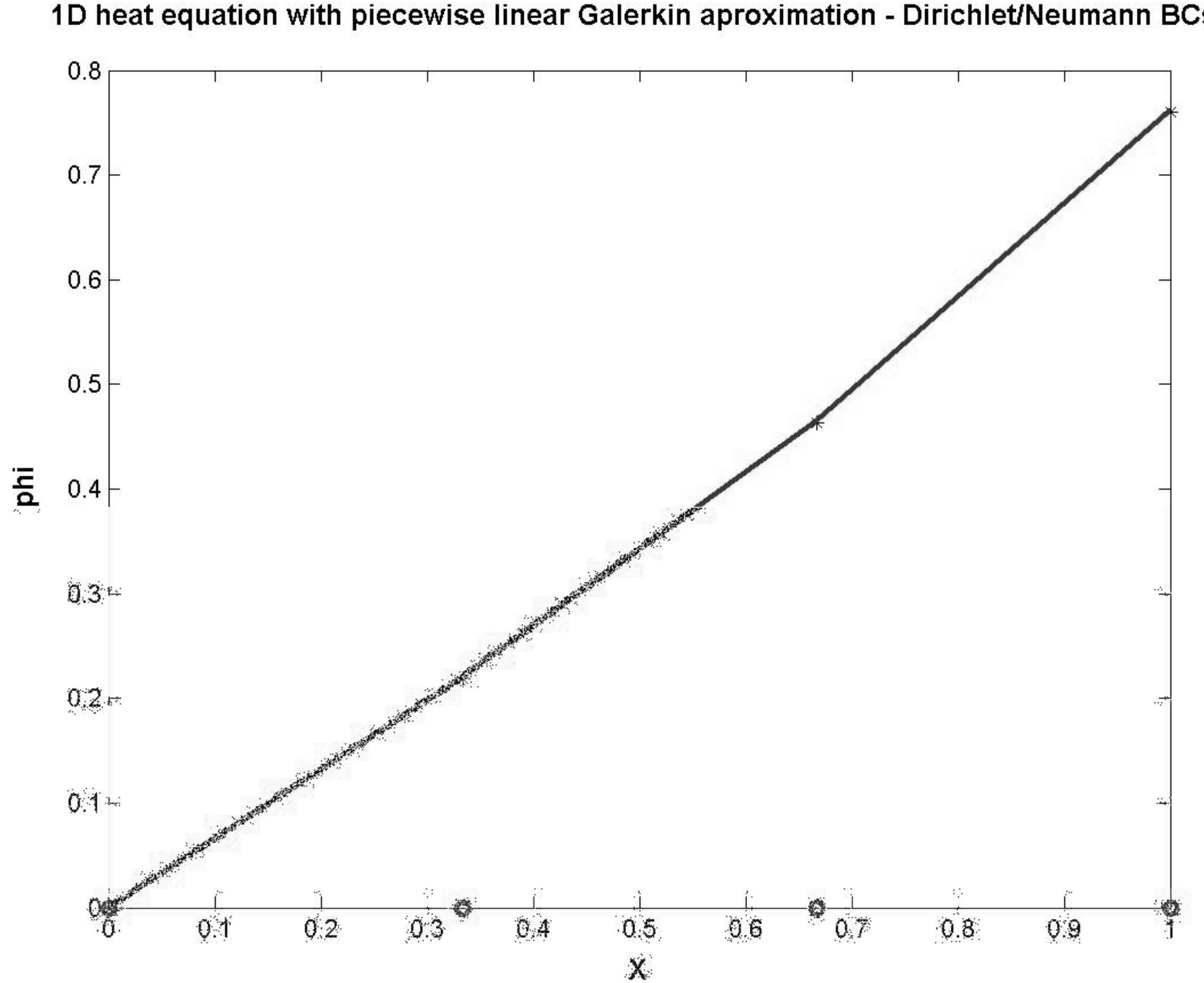
$$f(x=0) = A_1 e^{x} + A_2 e^{-x}\Big|_{x=0} = A_1 + A_2 = 0 \Rightarrow A_1 = -A_2$$

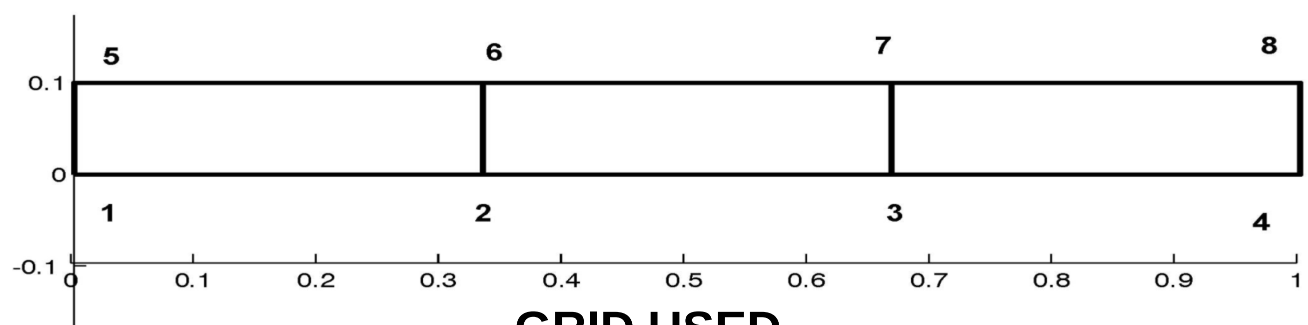
using $\dfrac{\mathrm{d}f}{\mathrm{dx}}(x=1) = 1$

$$\frac{\mathrm{d}f}{\mathrm{dx}}(x=1) = l_k A_k e^{l_k x}\Big|_{x=1} = A_1 e - A_2 e^{-1} = 1 \Rightarrow A_1 = \frac{1}{e + \frac{1}{e}}$$

$$\therefore \qquad f = \frac{1}{e + \frac{1}{e}}\left(e^{x} - e^{-x}\right) = \frac{e^{x} - e^{-x}}{e + \frac{1}{e}}$$

Example 5 Results with Neumann BC at x=1





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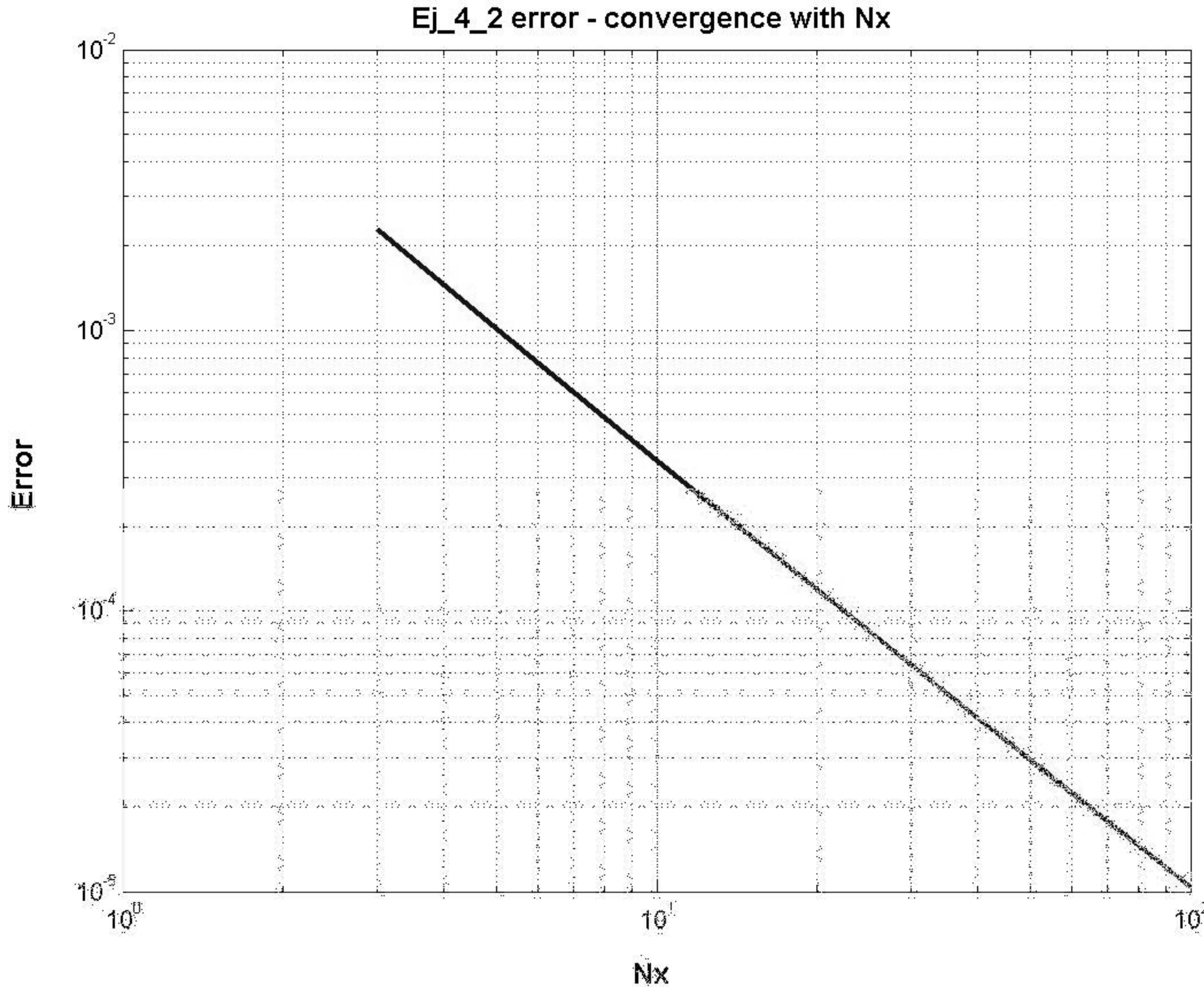
Example 5 Results with Neumann BC at x=1



[phi , phi_th] = […

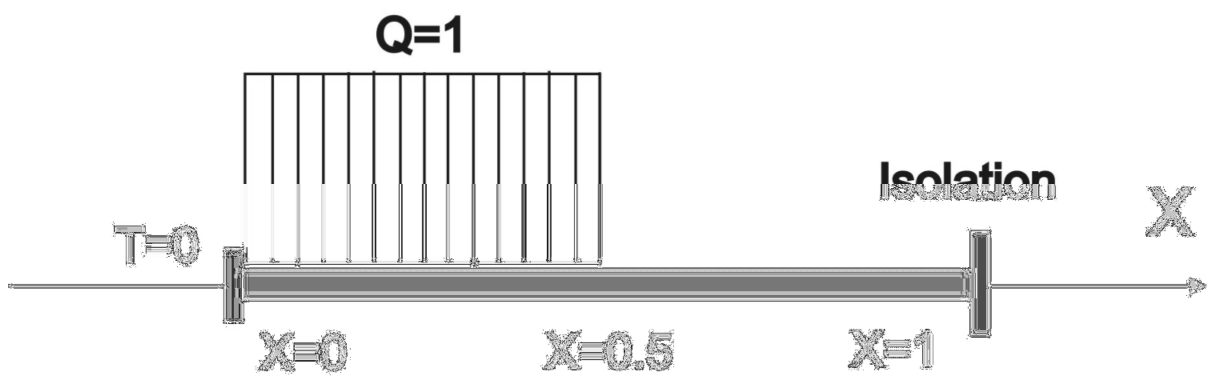
0	**0**
2.192827911335199e-001	**2.200407092120653e-001**
4.633853662097133e-001	**4.647576055524206e-001**
7.599367659842332e-001	**7.615941559557650e-001]**

Example 8 : SoDE – 2nd order to 1st order

$$\frac{d}{dx}\left(k\frac{df}{dx}\right)-Q=0$$

$$\Omega:\{x;0\le x\le 1\}\quad;\quad k=1\quad;\quad Q=\begin{cases}1 & x\le\frac{1}{2}\\ 0 & x>\frac{1}{2}\end{cases}$$

$$f=0\quad @\,x=0\quad;\quad q=-k\frac{df}{dx}=0\quad @\,x=1$$



Example 8 : two 1st order ODEs instead of one 2nd order ODE

Find $\hat{f}(x)$ solution of the following ODE $\frac{d}{dx}\left(k\,\frac{df}{dx}\right) - Q = 0$

as a 1st order system of ODEs with the vector of unknowns $\underline{f}^T = (q, f)$

the system to solve is $\begin{cases} q + k\,\frac{df}{dx} = 0 \\ \frac{dq}{dx} - Q = 0 \end{cases}$ in Ω

with the following approximation

$$\hat{q} = \sum_m q_m N_{m,q} \quad ; \quad \hat{f} = \sum_m f_m N_{m,f}$$

$$\therefore \int_0^1 N_{l,1}\, k\, \frac{d\hat{f}}{dx}\, dx + \int_0^1 N_{l,1}\, \hat{q}\, dx = 0 \qquad l = 1, 2, \cdots M_q$$

$$\int_0^1 N_{l,2}\, \frac{d\hat{q}}{dx}\, dx - \int_0^1 N_{l,2}\, Q\, dx = 0 \qquad l = 1, 2, \cdots M_f$$

Example 8 : SoDE – 2nd order to 1st order

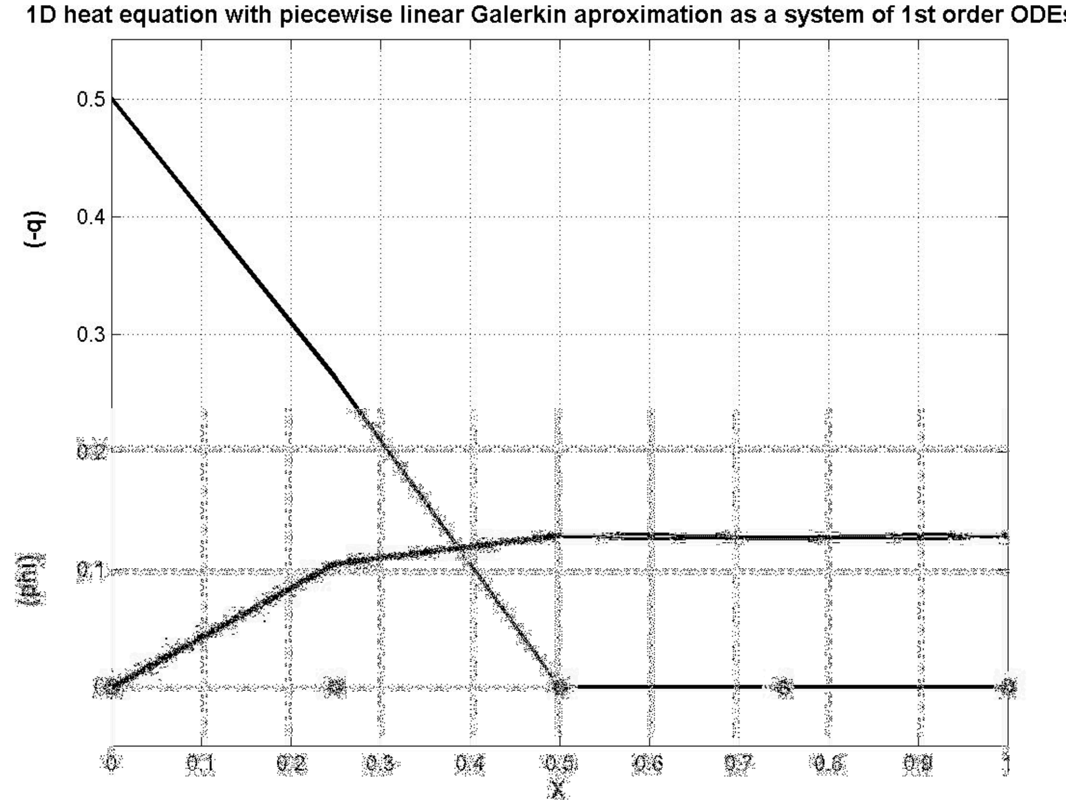
$$N_{l,1} = N_{l,2} = N_l \quad ; N_{m,q} = N_{m,f} = N_m$$

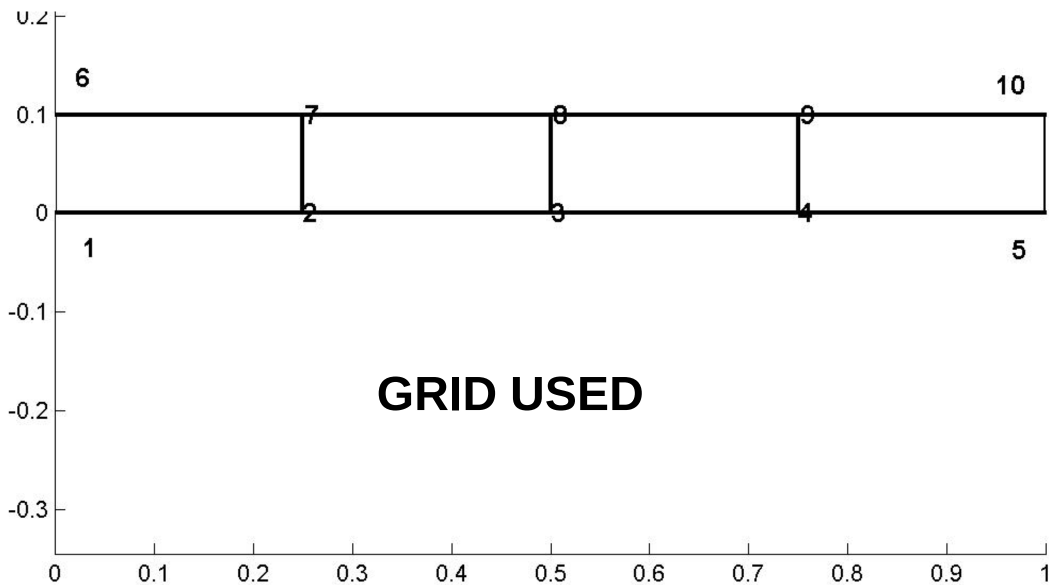
Contribution over each element

$$K^e = \int_0^1 \begin{bmatrix} N_{1,1}N_{1,q} & kN_{1,1}\dfrac{dN_{1,f}}{dx} & N_{1,1}N_{2,q} & kN_{1,1}\dfrac{dN_{2,f}}{dx} \\ N_{1,2}\dfrac{dN_{1,q}}{dx} & 0 & N_{1,2}\dfrac{dN_{2,q}}{dx} & 0 \\ N_{2,1}N_{1,q} & kN_{2,1}\dfrac{dN_{1,f}}{dx} & N_{2,1}N_{2,q} & kN_{2,1}\dfrac{dN_{2,f}}{dx} \\ N_{2,2}\dfrac{dN_{1,q}}{dx} & 0 & N_{2,2}\dfrac{dN_{2,q}}{dx} & 0 \end{bmatrix} dx$$

$$f^e = \int_0^1 \begin{bmatrix} 0 \\ N_{1,2}\,Q \\ 0 \\ N_{2,2}\,Q \end{bmatrix} dx$$

Example 8: Results





Example 8: Results from the book

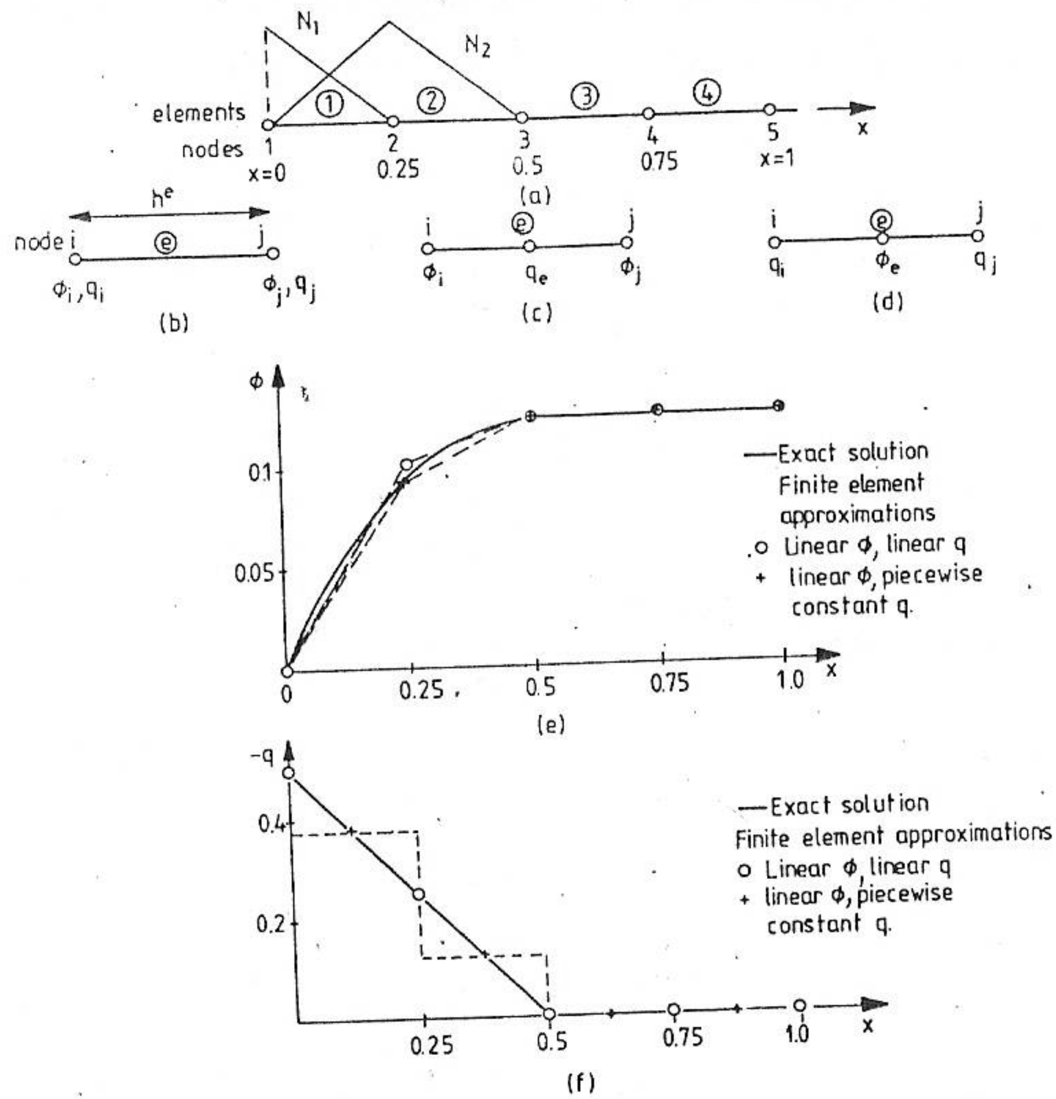


FIGURE 3.7. Solution of Example 3.3 showing (a) the node and element numbering system adopted with (b) piecewise linear ϕ and piecewise linear q. Also shown are the modifications which must be made for (c) piecewise linear ϕ, piecewise constant q, (d) piecewise constant ϕ, piecewise linear q. The graphs show a comparison of the approximations for (e) ϕ and (f) q.

Weak form and mixed formulation

Some figure with weight and trial
Functions for different fields

Example 8 : weak form and mixed formulation

Using a weak form for the second equation it is possible to change the approximation using piecewise constant for $\hat{q}$ and piecewise linear for $\hat{f} \Rightarrow M_f > M_q$

$$\hat{q} = \sum_{m=1}^{M_q} q_m N_{m,q} \quad ; \quad \hat{f} = \sum_{m=1}^{M_f} f_m N_{m,f}$$

$$\therefore \int_0^1 N_{l,1}\, k\, \frac{d\hat{f}}{dx}\, dx + \int_0^1 N_{l,1}\, \hat{q}\, dx = 0 \qquad l = 1,2,\cdots M_q$$

$$\int_0^1 N_{l,2}\, \frac{d\hat{q}}{dx}\, dx - \int_0^1 N_{l,2}\, Q\, dx =$$

$$-\int_0^1 \frac{dN_{l,2}}{dx}\hat{q} + \left[N_{l,2}\hat{q}\right]_{x=0}^{x=1} dx - \int_0^1 N_{l,2}\, Q\, dx = 0 \qquad l = 1,2,\cdots M_f$$

Weak form and mixed formulation

Contribution over each element

$$K^e = \int_0^{h^e} \begin{bmatrix} N_{1,1}N_{1,q} & kN_{1,1}\frac{dN_{1,f}}{dx} & N_{1,1}N_{2,q} & kN_{1,1}\frac{dN_{2,f}}{dx} \\ N_{1,2}\frac{dN_{1,q}}{dx} & 0 & N_{1,2}\frac{dN_{2,q}}{dx} & 0 \\ N_{2,1}N_{1,q} & kN_{2,1}\frac{dN_{1,f}}{dx} & N_{2,1}N_{2,q} & kN_{2,1}\frac{dN_{2,f}}{dx} \\ N_{2,2}\frac{dN_{1,q}}{dx} & 0 & N_{2,2}\frac{dN_{2,q}}{dx} & 0 \end{bmatrix} dx$$

at element level $M_q = 1$ (first equation) and $M_f = 2$ (second equation)

$\Rightarrow N_{2,1}$ does not exist anymore and $N_{2,q}$ either

$$\underline{\underline{K}}^e \underline{f}^e = \int_0^{h^e} \begin{bmatrix} N_{1,1}N_{1,q} & kN_{1,1}\frac{dN_{1,f}}{dx} & kN_{1,1}\frac{dN_{2,f}}{dx} \\ N_{1,2}\frac{dN_{1,q}}{dx} & 0 & 0 \\ N_{2,2}\frac{dN_{1,q}}{dx} & 0 & 0 \end{bmatrix} dx \begin{bmatrix} q_0 \\ f_1 \\ f_2 \end{bmatrix}$$

$$\underline{\underline{K}}^e = \int_0^{h^e} \begin{bmatrix} 0 & N_q\frac{dN_{1,2}}{dx} & 0 \\ kN_{2,1}\frac{dN_{1,f}}{dx} & N_{2,1}N_{2,q} & kN_{2,1}\frac{dN_{2,f}}{dx} \\ 0 & N_q\frac{dN_{2,2}}{dx} & 0 \end{bmatrix} dx \begin{bmatrix} f_1 \\ q_0 \\ f_2 \end{bmatrix}$$

Weak form and mixed formulation

$$\underline{\underline{K}}^e = \int_0^{h^e} \begin{bmatrix} 0 & N_q \dfrac{dN_{1,2}}{dx} & 0 \\ kN_{2,1} \dfrac{dN_{1,f}}{dx} & N_{2,1} N_{2,q} & kN_{2,1} \dfrac{dN_{2,f}}{dx} \\ 0 & N_q \dfrac{dN_{2,2}}{dx} & 0 \end{bmatrix} dx \begin{bmatrix} f_1 \\ q_0 \\ f_2 \end{bmatrix}$$

$$\underline{f}^e = -\int_0^{h^e} \begin{bmatrix} N_{1,2}\ Q \\ 0 \\ N_{2,2}\ Q \end{bmatrix} dx$$

for the first and the last element the rhs should be modified as

$$\underline{f}^{e=1} = -\int_0^{h^e} \begin{bmatrix} N_{1,2}\ Q \\ 0 \\ N_{2,2}\ Q \end{bmatrix} dx + \begin{bmatrix} -\hat{q}\big|_{x=0} \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{f}^{e=M_q} = -\int_0^{h^e} \begin{bmatrix} N_{1,2}\ Q \\ 0 \\ N_{2,2}\ Q \end{bmatrix} dx + \begin{bmatrix} 0 \\ 0 \\ \hat{q}\big|_{x=1} \end{bmatrix}$$

Mixed formulation for heat equation

```
for k=1:numel
    psi = (0:20)'/20;
    node1 = icone(k,1);
    node2 = icone(k,3);
    xx  = psi*(xnod(node2,1)-xnod(node1,1))+xnod(node1,1);
    eval(['Qx=' filename '(xx);' ]);
    [N,L_N] = shape_function(xx);

    Ke(k,1,2) = Ke(k,1,2) + trapz(xx,L_N(:,1));
    Ke(k,3,2) = Ke(k,3,2) + trapz(xx,L_N(:,2));
    Ke(k,2,1) = Ke(k,2,1) + kappa*trapz(xx,L_N(:,1));
    Ke(k,2,3) = Ke(k,2,3) + kappa*trapz(xx,L_N(:,2));
    Ke(k,2,2) = Ke(k,2,2) + 1*trapz(xx,ones(size(xx)));

    fe(k,1) = fe(k,1) - trapz(xx,Qx.*N(:,1));
    fe(k,3) = fe(k,3) - trapz(xx,Qx.*N(:,2));

end
```

Assembling

Mixed formulation for heat equation

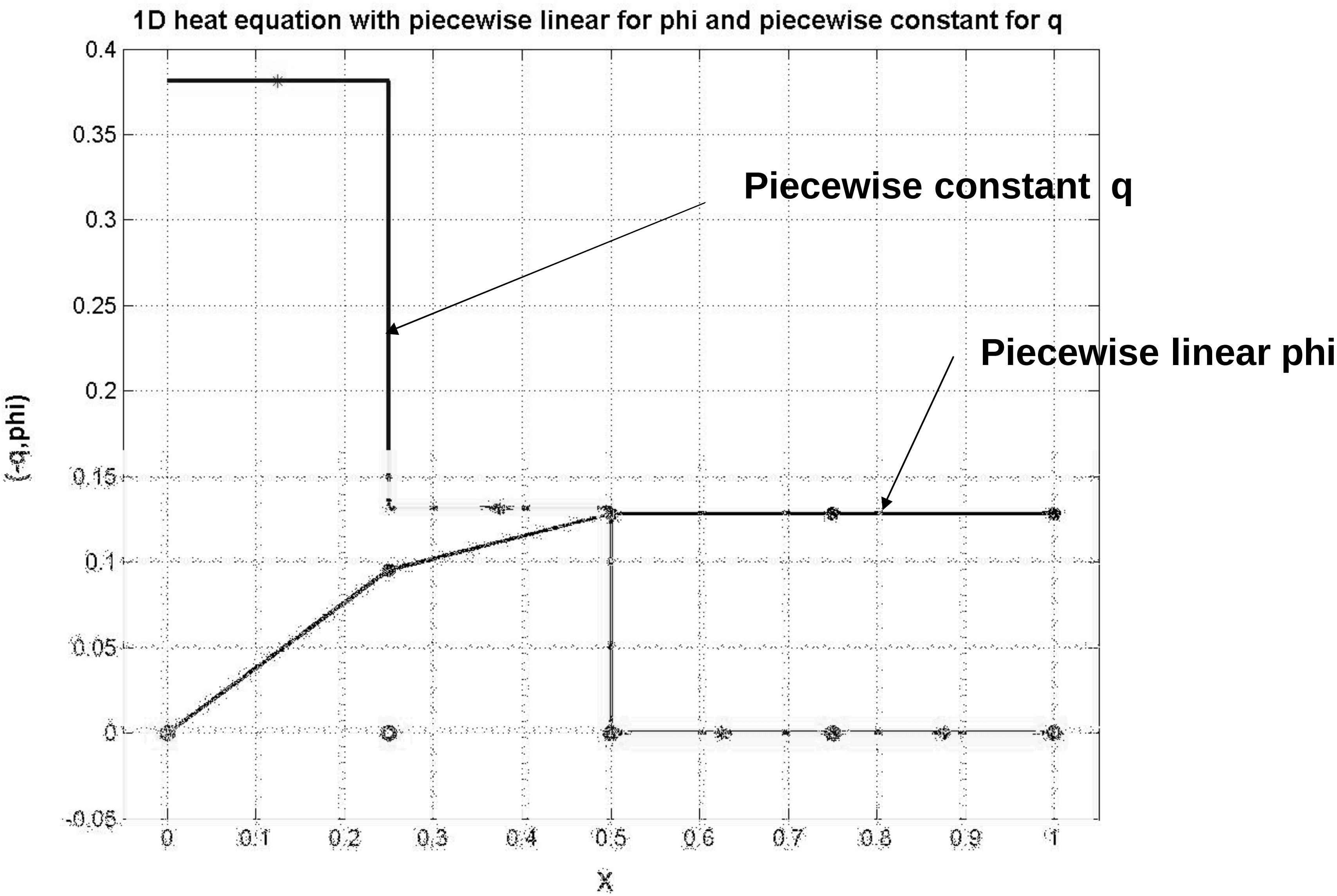
```
for k=1:numel
    node_i = icone(k,1);
    node_j = icone(k,3);
    node_e = icone(k,2);
    Kg(node_i,node_e)=Kg(node_i,node_e) + Ke(k,1,2);
    Kg(node_j,node_e)=Kg(node_j,node_e) + Ke(k,3,2);
    Kg(node_e,node_i)=Kg(node_e,node_i) + Ke(k,2,1);
    Kg(node_e,node_j)=Kg(node_e,node_j) + Ke(k,2,3);
    Kg(node_e,node_e)=Kg(node_e,node_e) + Ke(k,2,2);

    fg(node_i,1) = fg(node_i,1) + fe(k,1);
    fg(node_e,1) = fg(node_e,1) + fe(k,2);
    fg(node_j,1) = fg(node_j,1) + fe(k,3);
end
```

Gathering

```
% reducing the global system for fixations
free = (1:2*Nx+1)'; free(1) = [];
fg(free,1)    = fg(free,1) - Kg(free,1)    * phi_lef ;
Kg = Kg(free,free);
fg = fg(free,1);
% solver
phi_new = Kg\fg;
phi = [phi_lef;phi_new];
```

Solving

Mixed formulation - Results

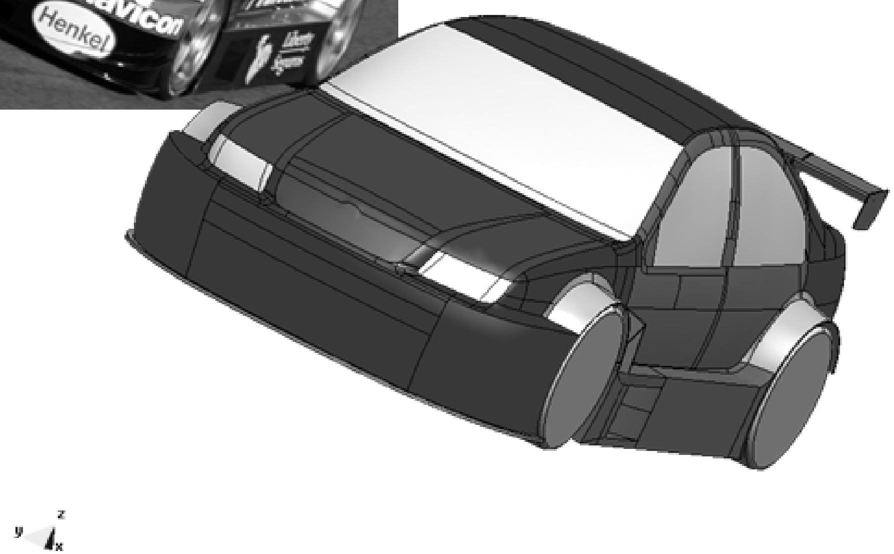


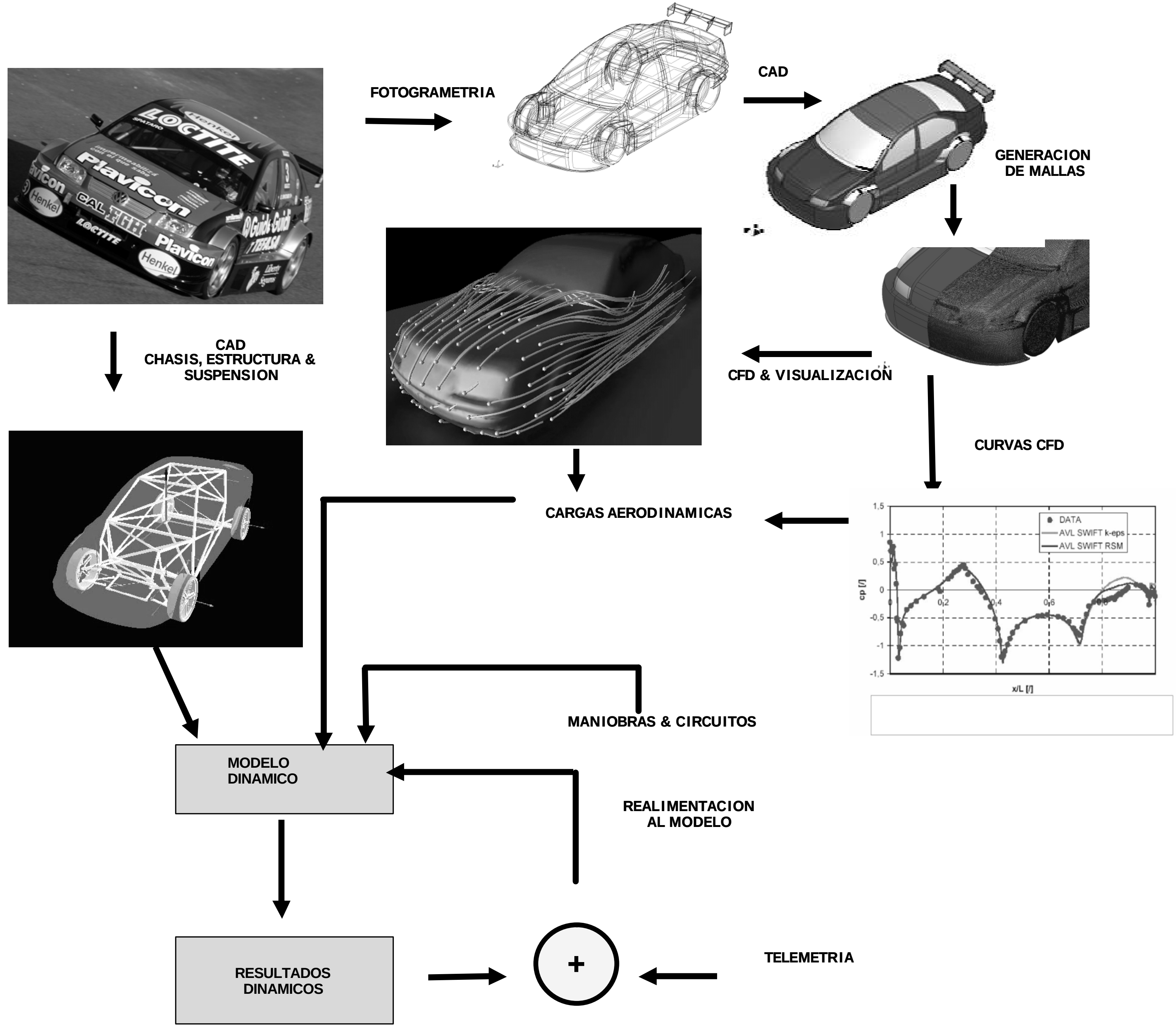
Generalization to 2D and 3D problems

- *1D problems are attractive to check numerical schemes (exact solution availability)*
- *1D problems have little practical (industrial) interests*
- *2D or 3D problems have analytical solution only for very simple geometries and boundary conditions*
- *In general only numerical methods may be used*
- *In 2D and 3D the choice of the shape functions depends on the equation to be solved and the type of element being used.*
- *Triangular and quadrangular types are commonly used (not necessarily restricted to these).*
- *In 3D tetrahedra are widely used because of mesh generators availability.*
- *Hexahedra are less common but they are attractive for their accuracy if the geometry allows for them.*
- *Prismatic elements close to bodies where boundary layer are expected may be included.*
- *2D & 3D the idea is to cover the whole domain with simple non-overlapping elements*

Example of 3D flow analysis

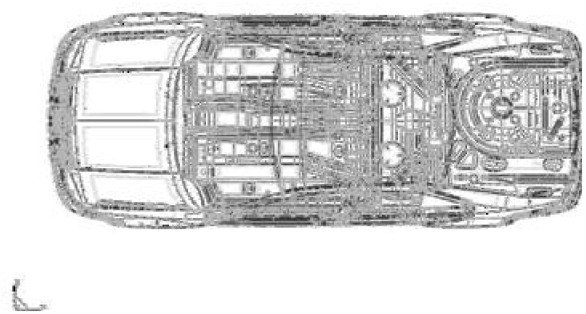
Aerodynamic analysis of a race car
The importance of computational geometry

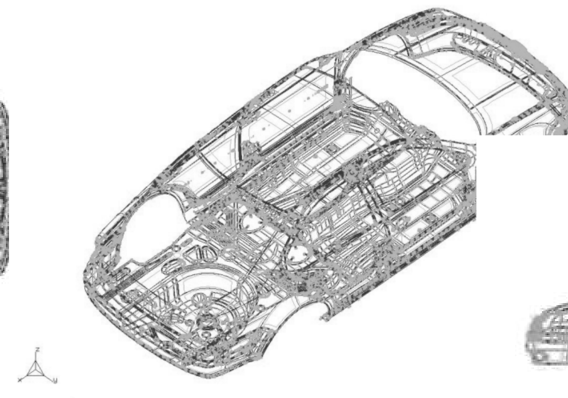


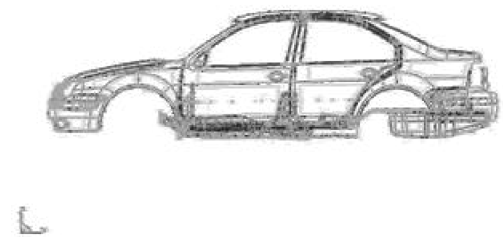

FOTOGRAMETRIA
CAD
GENERACION
DE MALLAS
CAD
CHASIS, ESTRUCTURA &
SUSPENSION
CFD & VISUALIZACION
CURVAS CFD
CARGAS AERODINAMICAS
DATA
AVL SWIFT k-eps
AVL SWIFT RSM
cp [/]
x/L [/]
MANIOBRAS & CIRCUITOS
MODELO
DINAMICO
REALIMENTACION
AL MODELO
RESULTADOS
DINAMICOS
+
TELEMETRIA

CAD – From production to analysis

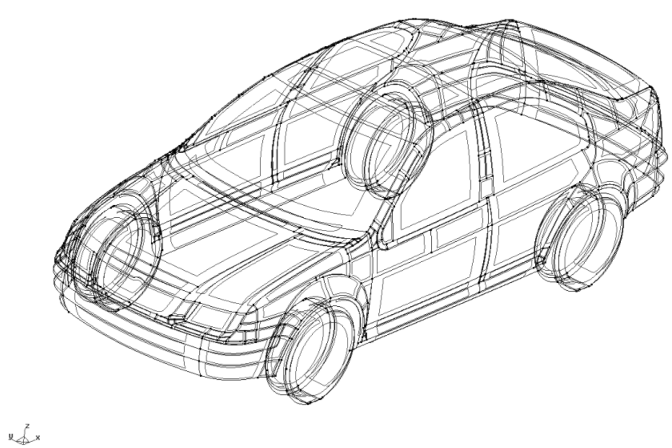
Geometría relevada

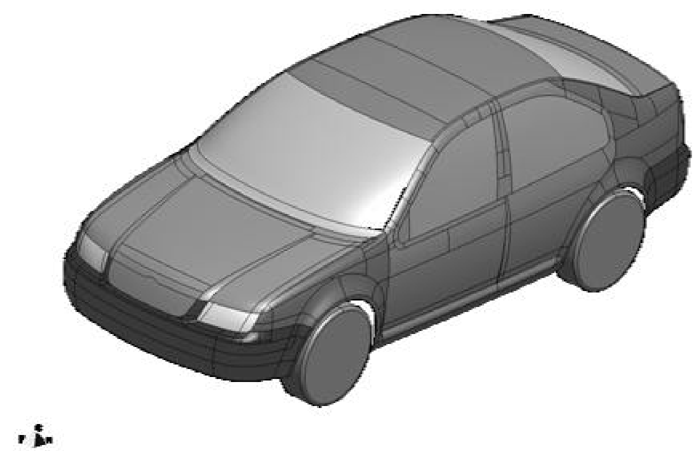




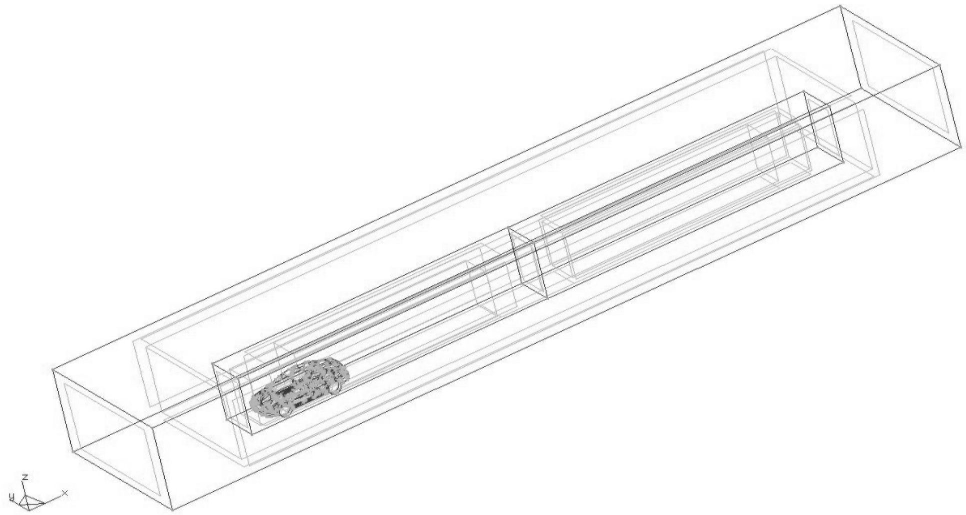


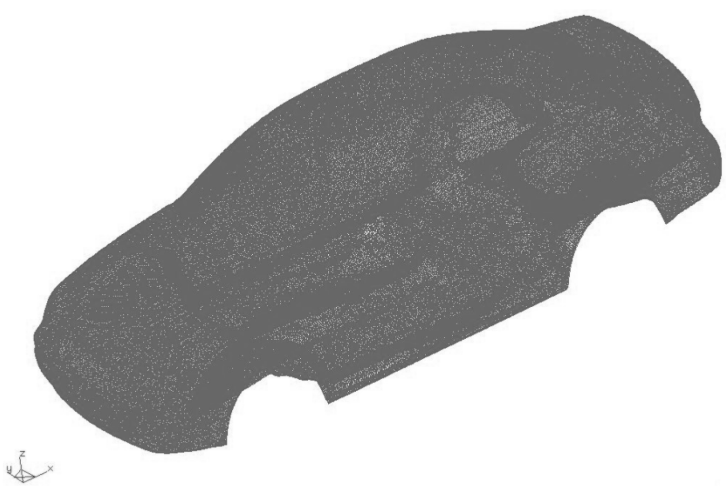
Geometría simplificada (limpia)

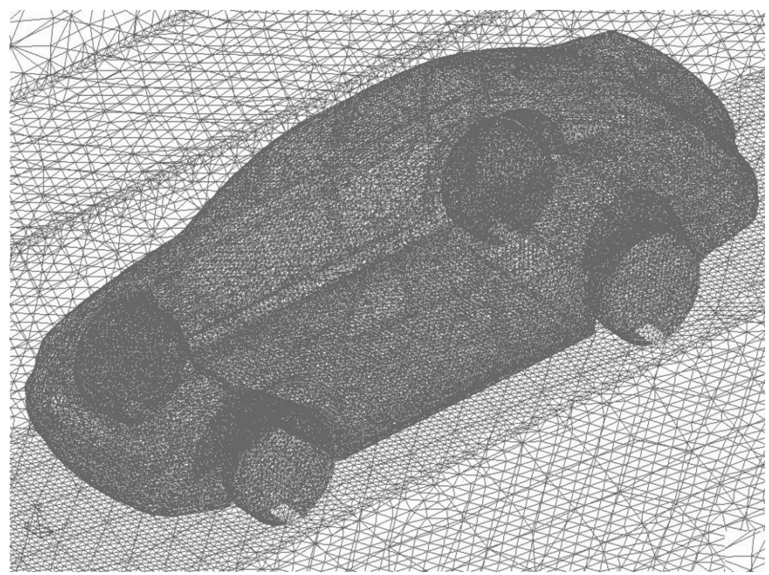




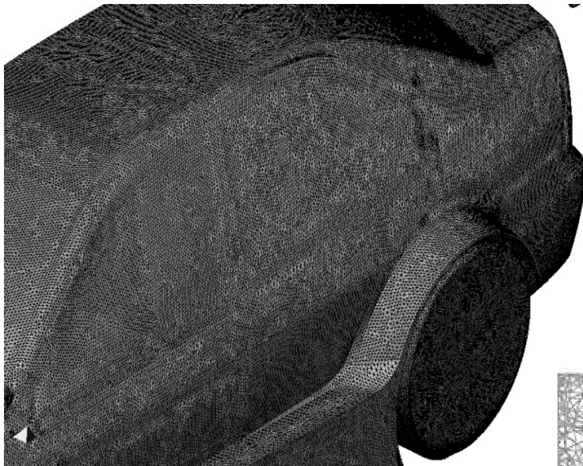
Geometrical model & mesh generation

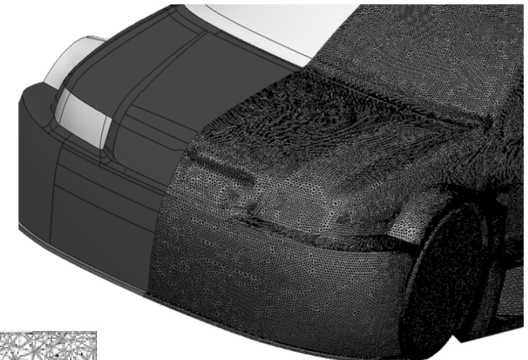


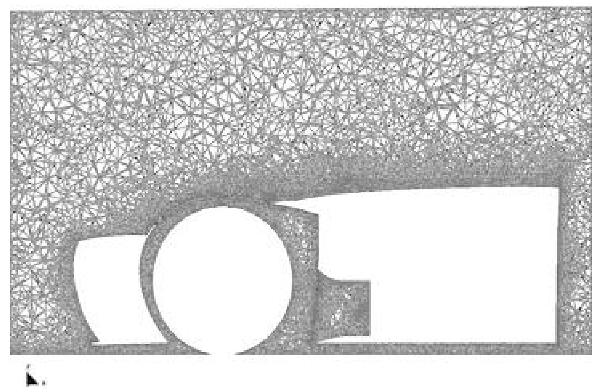


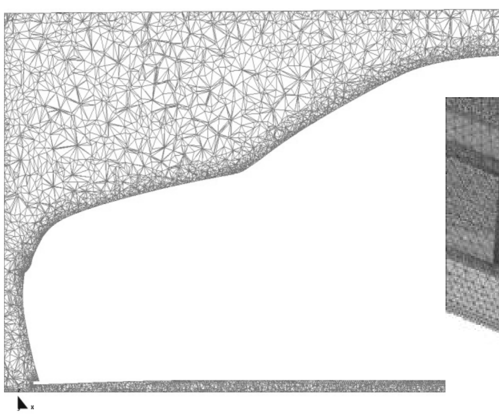


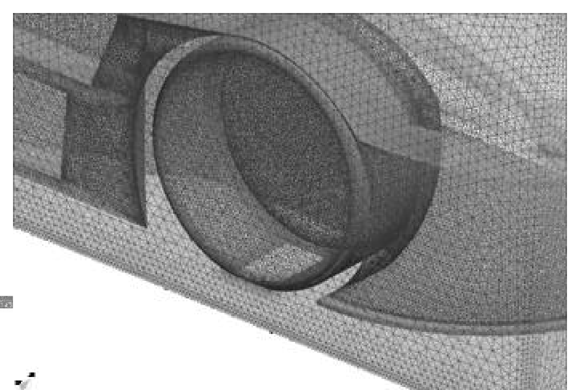
mesh generation details

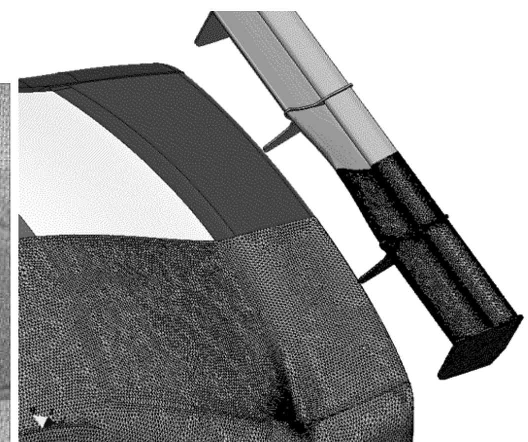












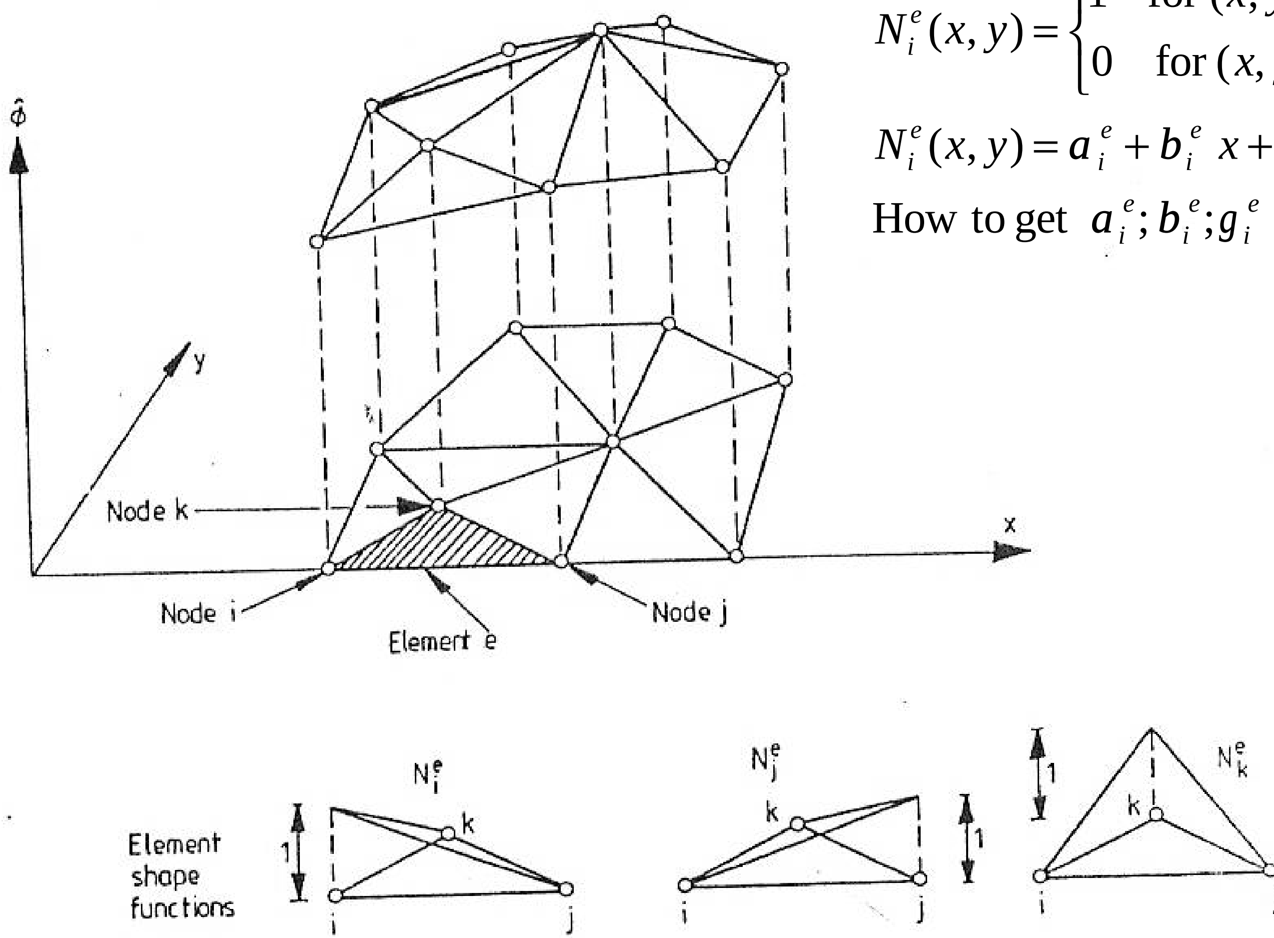
The linear triangle

Using piecewise linear over triangular elements

$$N_i^e(x,y)=\begin{cases}1 & \text{for } (x,y)=(x,y)_i\\ 0 & \text{for } (x,y)=(x,y)_j=(x,y)_k\neq(x,y)_i\end{cases}$$

$$N_i^e(x,y)=\boldsymbol{a}_i^e+\boldsymbol{b}_i^e\,x+\boldsymbol{g}_i^e\,y \qquad \text{on element } e$$

How to get $\boldsymbol{a}_i^e;\boldsymbol{b}_i^e;\boldsymbol{g}_i^e$?



(b)

FIGURE 3.2. (*continued*).

The linear triangle

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_i^e \\ b_i^e \\ g_i^e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with solution

$$a_i^e = \frac{x_j y_k - x_k y_j}{2\Delta^e}$$

$$b_i^e = \frac{y_j - y_k}{2\Delta^e}$$

$$g_i^e = \frac{x_k - x_j}{2\Delta^e}$$

$$2\Delta^e = \det \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} = 2A^e = 2(\text{area of element } e)$$

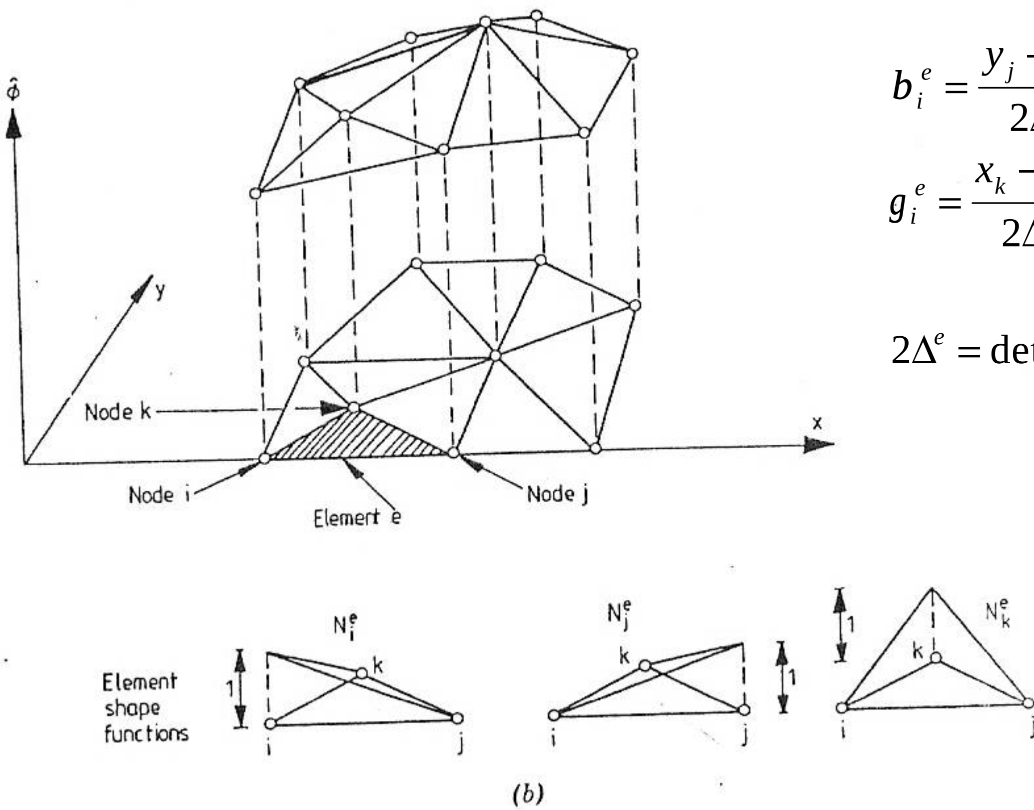


FIGURE 3.2. (*continued*).

The linear triangle – How the numbering impact on the algebraic structure of the stiffness matrix ?

Boundaries approximated by straight lines

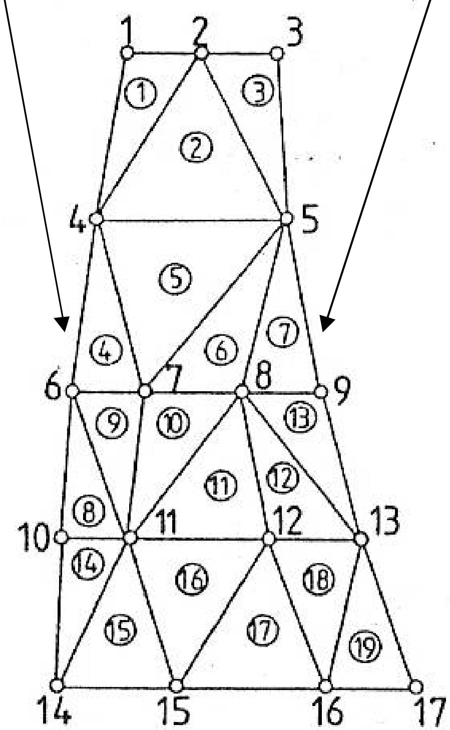


FIGURE 3.10. Triangular finite elements used to represent the profile of a dam.

*** Non zeros terms**

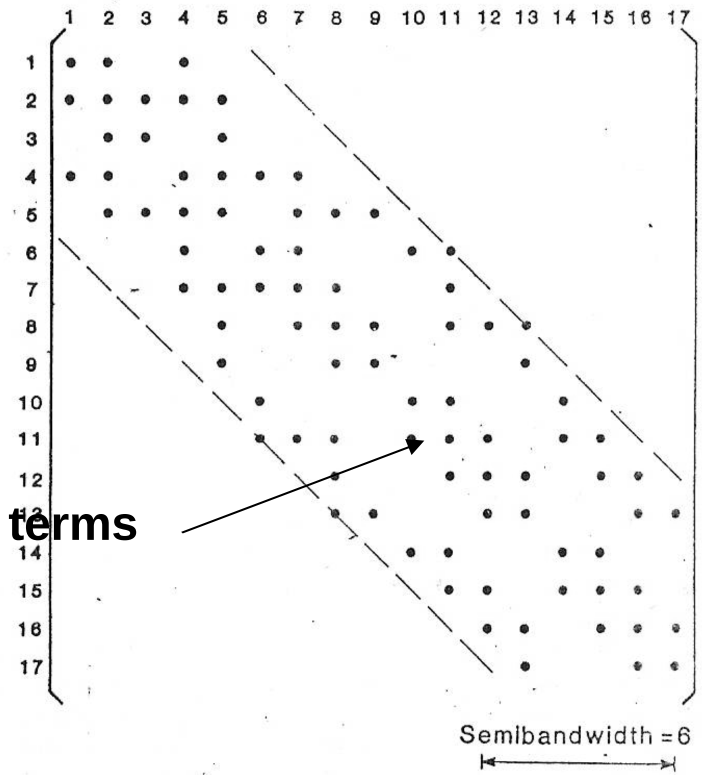


FIGURE 3.11. Position of the nonzero entries in the matrix K resulting from the assembly of the contributions from the elements shown in Fig. 3.10.

The linear triangle – How the numbering impact on the algebraic structure of the stiffness matrix ?

- **The numbering influences the bandwidth**
- **The bandwidth is closely related with the computational cost and memory requirements.**
- **Sparse matrices are found due to the narrow-based trial functions**
- **The computation may be divided in three main steps**
  - **Element assembling**
  - **Gathering**
  - **Solver**

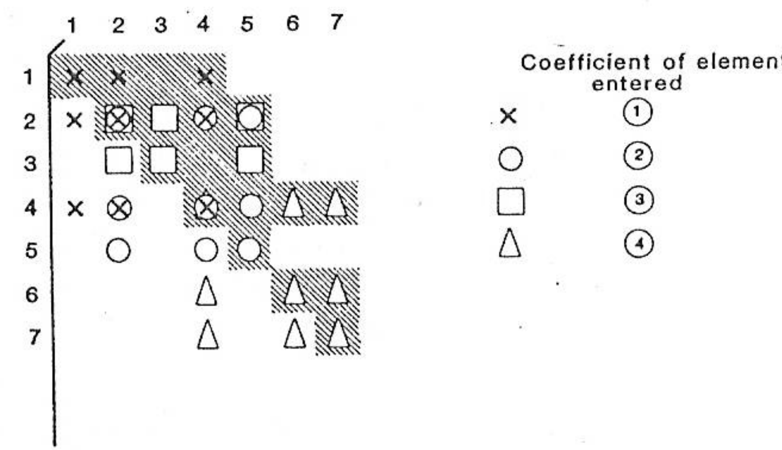


FIGURE 3.12. Process of assembly of elements 1–4 in the problem of Fig. 3.10. The shaded area represents the information that must be stored if the matrix is symmetric.

The linear triangle – How the numbering impact on the algebraic structure of the stiffness matrix ?

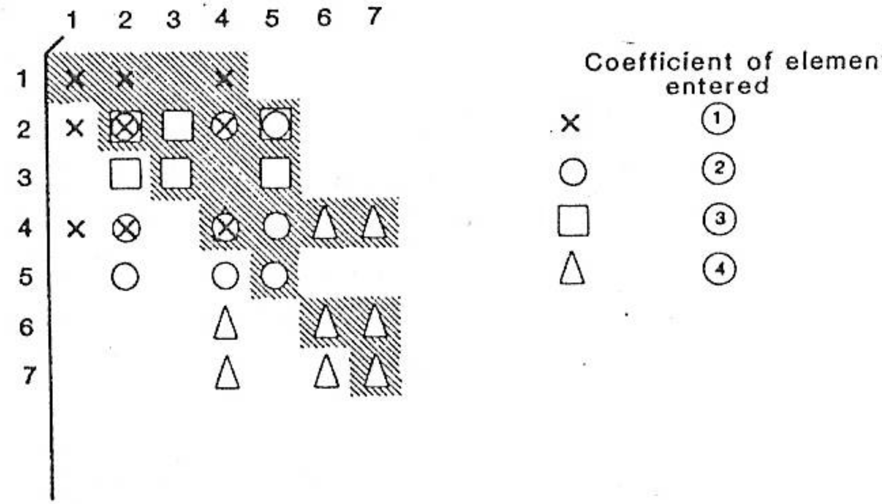


FIGURE 3.12. Process of assembly of elements 1–4 in the problem of Fig. 3.10. The shaded area represents the information that must be stored if the matrix is symmetric.

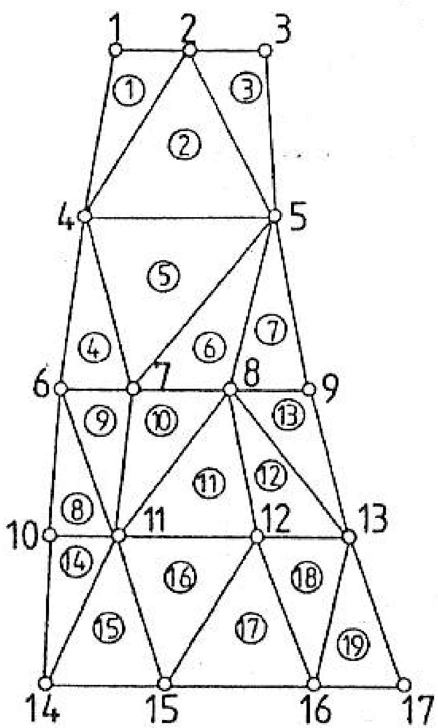


FIGURE 3.10. Triangular finite elements used to represent the profile of a dam.

The bilinear rectangle

A rectangle is a sort of cartesian product of linear segments

The cartesian product of linear trial functions in 1D

produces a bilinear 2D or a trilinear 3D shape functions

$$N_i^e(x,y) = \frac{(h_x - x)}{h_x}\frac{(h_y - y)}{h_y}$$

$$\therefore \quad N_i^e(x,y) = \begin{cases} 1 & \text{for } (x,y) = (x,y)_i \\ 0 & \text{for } (x,y) = (x,y)_j = (x,y)_k = (x,y)_l \neq (x,y)_i \end{cases}$$

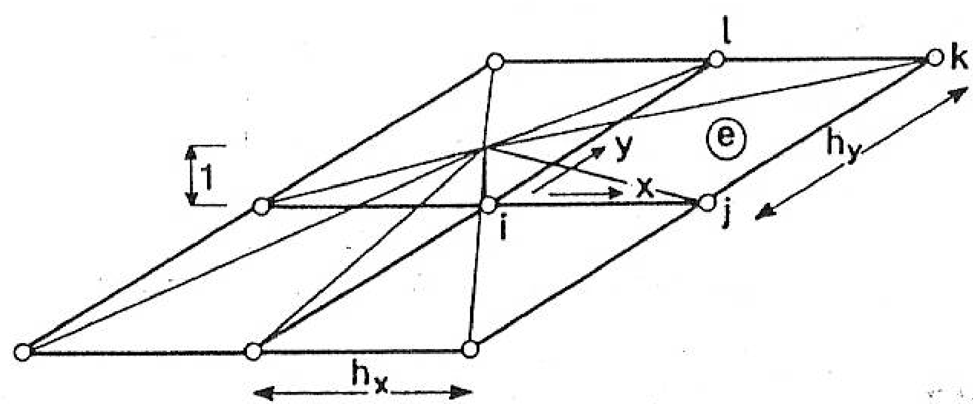


FIGURE 3.13. Bilinear shape function associated with node *i* of a typical four-noded rectangular element.

The bilinear rectangle

$$N_i^e(x,y) = a_i^e + b_i^e\ x + g_i^e\ y\ + d_i^e\ xy \quad \text{on element } e$$

How to get $a_i^e; b_i^e; g_i^e; d_i^e$?

$$\begin{bmatrix} 1 & x_i & y_i & x_i y_i \\ 1 & x_j & y_j & x_j y_j \\ 1 & x_k & y_k & x_k y_k \\ 1 & x_l & y_l & x_l y_l \end{bmatrix} \begin{bmatrix} a_i^e \\ b_i^e \\ g_i^e \\ d_i^e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Low order (linear) and
low regularity (C0) of shape function

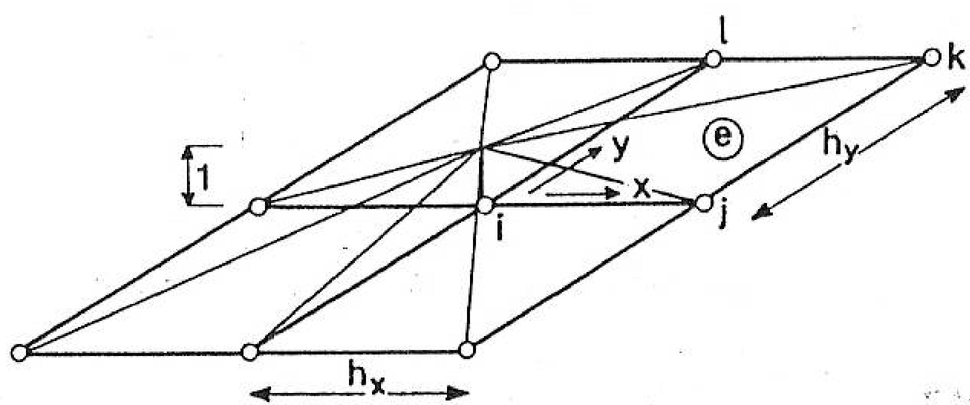


FIGURE 3.13. Bilinear shape function associated with node *i* of a typical four-noded rectangular element.

Approximation of curved boundaries

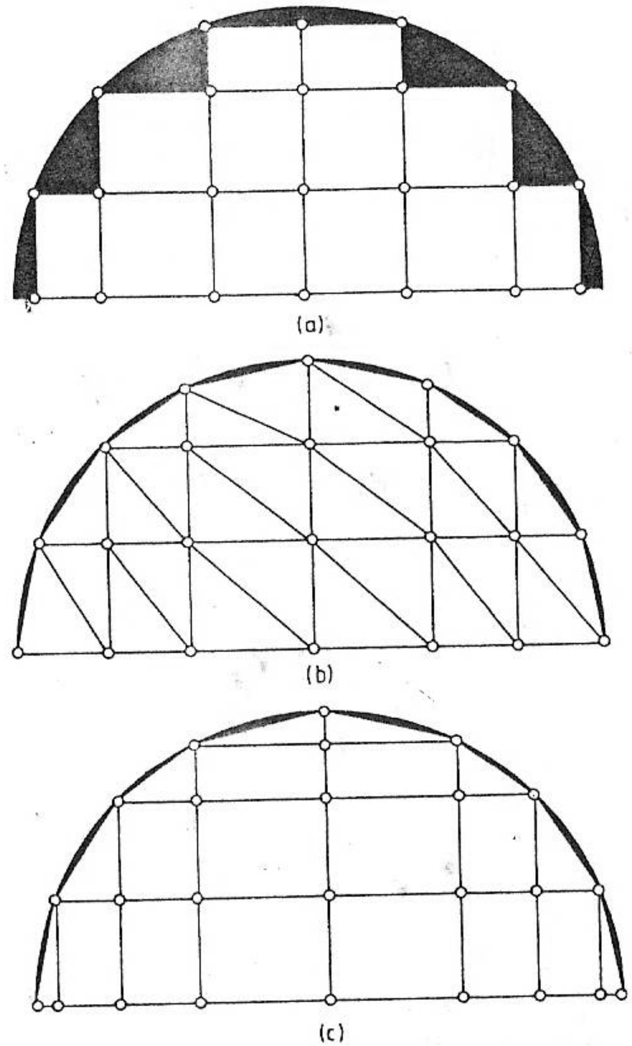


FIGURE 3.14. Finite element subdivision of a semicircular region using (*a*) rectangular elements only, (*b*) triangular elements with the same total number of nodes, (*c*) a combination of triangular and rectangular elements with additional boundary nodes. In each case the dark areas indicate the magnitude of the error made in the representation of the domain.

3D elements – tetrahedron and hexahedron

$$N_i^e(x,y,z) = \begin{cases} 1 & \text{for } (x,y,z) = (x,y,z)_i \\ 0 & \text{for } (x,y,z) = (x,y,z)_j = (x,y,z)_k = (x,y,z)_l \neq (x,y,z)_i \end{cases}$$

$$N_i^e(x,y,z) = a_i^e + b_i^e\, x + g_i^e\, y + d_i^e\, z \quad \text{on element } e$$