EJEMPLO Aplicación Prin. Vas a Ecuación Elasticidad

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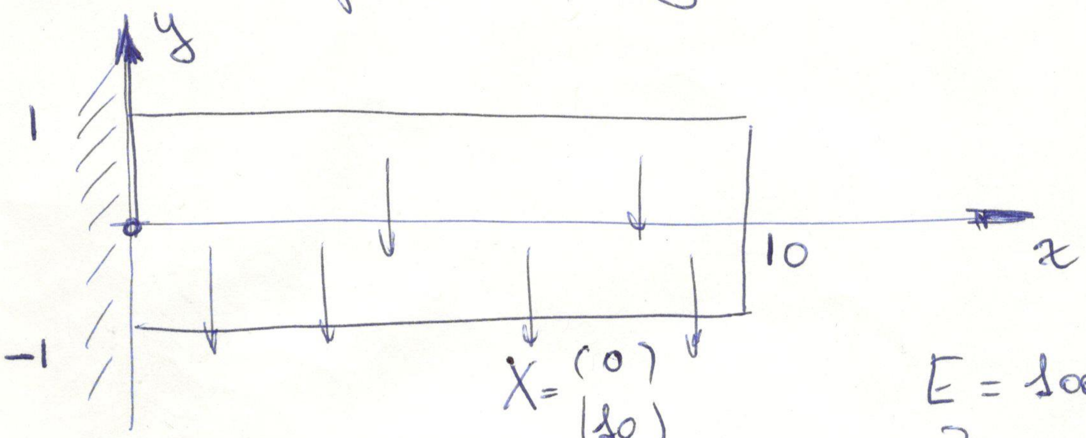
P.T.V.

$$\int_V X_i\, \delta u_i\, dV + \int_{S_\sigma} \overset{\nu}{T_i}\, \delta u_i\, dS = \int_V \sigma_{ij}\, \delta e_{ij}\, dV$$

Sea el problema siguiente:



$E = 1000$

$\nu = 0.3$

Material Isótropo

Aproximamos

$$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & & \underline{0} & \\ & & \underline{0} & & & & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{bmatrix} \cancel{U_{00}} \\ U_{01} \\ \cancel{U_{11}} \\ U_{02} \\ U_{12} \\ \cancel{U_{22}} \\ V_{00} \\ V_{01} \\ V_{11} \\ \vdots \\ V_{22} \end{bmatrix}$$

Notar que debe ser

$$u(x=0,y)=0$$

Luego

$$u = U_{00} + \cancel{x U_{01}} + y U_{11} + \cancel{x^2 U_{02}} + \cancel{xy U_{12}} + y^2 U_{22}$$

$$= U_{00} + y U_{11} + y^2 U_{22} = 0$$

$$\Rightarrow \quad U_{00} = 0, \quad U_{11} = 0, \quad U_{22} = 0$$

$$v(x=0,y) = 0$$

$$v = V_{00} + x V_{01} + \dots = 0$$

$$\Rightarrow \quad V_{00} = 0, \quad V_{11} = 0, \quad V_{22} = 0$$

Luego, diremos:

$$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{bmatrix} x & x^2 & xy & 0 & 0 & 0 \\ 0 & 0 & 0 & x & x^2 & xy \end{bmatrix}}_{\underline{\underline{N}}} \begin{pmatrix} U_{10} \\ U_{20} \\ U_{11} \\ V_{10} \\ V_{20} \\ V_{11} \end{pmatrix}$$

$$\therefore \quad \delta \underline{u} = \underline{\underline{N}}\, \delta \underline{U}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Lo escribimos vectorialmente:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\lfloor x \;\; x^2 \;\; xy \rfloor \begin{pmatrix} U_{10} \\ U_{20} \\ U_{11} \end{pmatrix} \\ \frac{\partial}{\partial y}\lfloor x \;\; x^2 \;\; xy \rfloor \begin{pmatrix} V_{10} \\ V_{20} \\ V_{11} \end{pmatrix} \\ \frac{\partial}{\partial y}\lfloor x \;\; x^2 \;\; xy \rfloor \begin{pmatrix} U_{10} \\ U_{20} \\ U_{11} \end{pmatrix} + \frac{\partial}{\partial x}\lfloor x \;\; x^2 \;\; xy \rfloor \begin{pmatrix} V_{10} \\ V_{20} \\ V_{11} \end{pmatrix} \end{pmatrix}$$

Trabajando:

$$= \underbrace{\begin{bmatrix} 1 & 2x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & x & 1 & 2x & y \end{bmatrix}}_{\underline{\underline{B}}} \begin{pmatrix} U_{10} \\ U_{20} \\ U_{11} \\ V_{10} \\ V_{20} \\ V_{11} \end{pmatrix}$$

Para un material isótropo

$$\sigma_{ij} = \lambda e_{\alpha\alpha} \delta_{ij} + 2G e_{ij}$$

En estado plano de tensiones:

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(e_{xx} + \nu e_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(e_{yy} + \nu e_{xx}\right)$$

$$\sigma_{xy} = \frac{E}{1+\nu} e_{xy}$$

Luego, ~~Deform~~ matricialmente:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \underbrace{E \begin{bmatrix} \frac{1}{1-\nu^2} & \frac{\nu}{1-\nu^2} & 0 \\ \frac{\nu}{1-\nu^2} & \frac{1}{1-\nu^2} & 0 \\ 0 & 0 & \frac{1}{2(1+\nu)} \end{bmatrix}}_{\underline{\underline{C}}} \begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix}$$

En consecuencia:

$$\underline{\varepsilon} = \begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix} = \underline{\underline{B}}\ \underline{U} \Rightarrow \delta\underline{\varepsilon} = \underline{\underline{B}}\ \delta\underline{U}$$

$$\underline{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \underline{\underline{C}}\ \underline{\varepsilon}$$

$$\sigma_{ij}\, \delta e_{ij} = \sigma_{11}\, \delta e_{11} + \sigma_{12}\, \delta e_{12} + \sigma_{22}\, \delta e_{22} + \sigma_{21}\, \delta e_{21} =$$

$$= \underline{\sigma} \cdot \delta\underline{\varepsilon}$$

Nota el 2!

Así, el PTV

$$\int_V \delta \underline{u}^T \underline{X}\, dV = \int_V \delta \underline{e}^T \underline{\sigma}\, dV$$

$$\int_V \delta U^T \underline{\underline{N}}^T \underline{X}\, dV = \int_V \delta \underline{U}^T \underline{\underline{B}}^T \underline{\underline{C}}\, \underline{e}\, dV =$$

$$= \int_V \delta \underline{U}^T \underline{\underline{B}}^T \underline{\underline{C}}\, \underline{\underline{B}}\, \underline{U}\, dV$$

$\delta \underline{U}$, U sale del signo integral:

$$\cancel{\delta U^T} \int_V \underline{\underline{N}}^T \underline{X}\, dV = \cancel{\delta U^T} \int \underline{\underline{B}}^T \underline{\underline{C}}\, \underline{\underline{B}}\, dV\, \underline{U}$$

$$\therefore \quad \int_V \underline{\underline{B}}^T \underline{\underline{C}}\, \underline{\underline{B}}\, dV\, \underline{U} = \int \underline{\underline{N}}^T \underline{X}\, dV$$

$$\boxed{\underline{\underline{K}}\ \underline{U} = \underline{F}}$$