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10 DERIVATION OF FIELD EQUATIONS

In the preceding chapters, we have analyzed deformation (strain) and flow (strain rate) and their relationship with the force of interaction (stress) between parts of a material body (continuum). We are now in a position to use this information to derive differential equations describing the motion of the continuum under specific boundary conditions. Our formulation must obey Newton's law of motion, the principle of conservation of mass, and the laws of thermodynamics. This chapter is concerned with expressing these laws in a form suitable for the treatment of a continuum.

One may wonder why there is a need for further elaboration on these well-known laws. The answer may be illustrated in the following example. If we have a single particle, the principle of conservation of mass merely states that the mass of the particle is a constant. However, if we have a large number of particles, such as the water droplets in a cloud, the situation requires some thought. For the cloud, it is no longer practical to identify the individual particles. The most convenient way to describe the cloud is to consider the velocity field, the density distribution, the temperature distribution, etc. It is the description of the classical conservation laws in such a circumstance that will occupy our attention in this chapter.

Our approach is based on the fact that these conservation laws must be applicable to the matter enclosed in a volume bounded by an arbitrary closed surface. In such an approach, we find that some quantities enter naturally in a surface integral, others in a volume integral. A transformation from a surface integral to a volume integral, and vice versa, is often required. This transformation is embodied in Gauss's theorem, which serves as our mathematical starting point.

10.1 GAUSS'S THEOREM

We shall begin with the derivation of Gauss's theorem. Consider a convex region V bounded by a surface S that consists of a finite number of parts whose outer normals form a continuous vector field (e.g., the one shown in Fig. 10.1). Such a

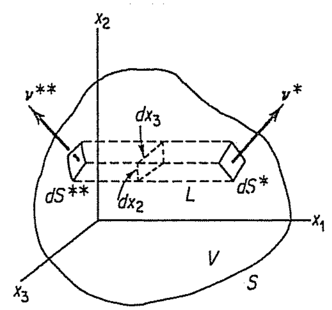

Figure 10.1 Path of integration illustrating the derivation of Gauss's theorem.

region is said to be *regular*. Let a function $A(x_1, x_2, x_3)$ be defined in the volume V and on the surface S. Let A be continuously differentiable in V. Let us consider the volume integral

$$\iiint_V \frac{\partial A}{\partial x_1}\, dx_1\, dx_2\, dx_3.$$

The integrand is the partial derivative of A with respect to x_1. By integrating with respect to x_1 along a line segment L, we obtain

$$\iiint_V \frac{\partial A}{\partial x_1}\, dx_1\, dx_2\, dx_3 = \iint_S (A^* - A^{**})\, dx_2\, dx_3 \qquad (10.1\text{–}1)$$

where A^* and A^{**} are, respectively, the values of A on the surface S at the right and left ends of the line segment L parallel to the x_1-axis. The surface integral on the right-hand side of Eq. (10.1–1) may be written more elegantly. The factors $+dx_2\, dx_3$ and $-dx_2\, dx_3$ are the projections on the x_2x_3-plane of the areas dS^* and dS^{**} at the ends of the line segment L. Let $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ be the unit vector along the outer normal to the surface S. For the element shown in Fig. 10.1, we see that $\nu_1^* = \cos(x_1, \boldsymbol{\nu}^*)$ is positive, whereas $\nu_1^{**} = \cos(x_1, \boldsymbol{\nu}^{**})$ is negative. It is easy to see that in this case, $dx_2\, dx_3 = \nu_1^*\, dS^*$ at the right end and $-dx_2\, dx_3 = \nu_1^{**}\, dS^{**}$ at the left end. Therefore, the surface integral in Eq. (10.1–1) can be written as

$$\iint_S (A^*\, dx_2\, dx_3 - A^{**}\, dx_2\, dx_3) = \iint_S (A^*\nu_1^*\, dS^* + A^{**}\nu_1^{**}\, dS^{**}). \qquad (10.1\text{–}2)$$

The asterisks may be omitted because they merely indicate the appropriate values of A and ν_1 to be taken in a surface integral according to conventional notations. Thus, the right-hand side of Eq. (10.1–1) reduces to $\int_S A\nu_1\, dS$. Now, if we write the volume integral on the left-hand side as $\int_V (\partial A/\partial x_1)\, dV$, then we have

$$\int_V \frac{\partial A}{\partial x_1}\, dV = \int_S A\nu_1\, dS, \qquad (10.1\text{–}3)$$

where dV and dS denote the elements of V and S, respectively. A similar argument applies to the volume integral of $\partial A/\partial x_2$ or $\partial A/\partial x_3$. Thus, we obtain Gauss's theorem,

$$\int_V \frac{\partial A}{\partial x_i}\, dV = \int_S A\nu_i\, dS, \qquad (i = 1, 2, 3). \qquad ▲\ (10.1\text{–}4)$$

This formula holds for any convex regular region or for any region that can be decomposed into a finite number of convex regular regions.

Now let us consider a tensor field $A_{jkl\ldots}$. Let the region V with boundary surface S be within the region of definition of $A_{jkl\ldots}$. Let every component of $A_{jkl\ldots}$ be continuously differentiable in V. Then Eq. (10.1–4) is applicable to every component of the tensor, and we obtain the general result

$$\int_V \frac{\partial}{\partial x_i} A_{jkl\ldots}\, dV = \int_S \nu_i A_{jkl\ldots}\, dS, \qquad (10.1\text{–}5)$$

which is one of the most useful theorems in applied mathematics.

This theorem was given in various forms by Lagrange (1762), Gauss (1813), Green (1828), and Ostrogradsky (1831). It is best known in this country as *Green's theorem* or *Gauss's theorem*.

Example 1

Let ν_i represent a vector. Then, according to Eq. (10.1–5), we have, on identifying $A_i = \nu_i$, and n_i as the normal vector to the surface S,

$$\int_V \frac{\partial \nu_i}{\partial x_i}\, dV = \int_S \nu_i n_i\, dS. \qquad (10.1\text{–}6)$$

If we write the coordinates x_1, x_2, x_3 as x, y, z; the components ν_1, ν_2, ν_3 as u, v, w; and the direction cosines n_1, n_2, n_3 of the outer normal to the surface S as l, m, n, then

$$\iiint_V \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\, dy\, dz = \iint_S (lu + mv + nw)\, dS. \qquad (10.1\text{–}7)$$

In another popular notation, we denote the vector by $\mathbf{v}$ and the scalar product $\nu_i n_i$ by $\mathbf{v}\cdot\mathbf{n}$ and define

$$\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \qquad (10.1\text{–}8)$$

Then Eq. (10.1–7) becomes

$$\int_V \operatorname{div} \mathbf{v}\, dV = \int_S \mathbf{v}\cdot\mathbf{n}\, dS. \qquad ▲\ (10.1\text{–}9)$$

Equations (10.1–6), (10.1–7), and (10.1–9) are the best known forms of Gauss's theorem.

Example 2

If A is identified with a potential function ϕ, then Eq. (10.1–3) is usually written in the vector form

$$\int_V \text{grad } \phi \, dV = \int_S \mathbf{n}\phi \, dS.$$

Example 3

Let e_{ijk} be the permutation tensor. Then

$$\int e_{ijk}u_{k,j}dV = e_{ijk}\int u_{k,j}dV = e_{ijk}\int u_k n_j dS = \int e_{ijk}u_k n_j dS;$$

i.e.,

$$\int \text{curl } \mathbf{u} \, dV = \int \mathbf{n} \times \mathbf{u} \, dS.$$

10.2 MATERIAL DESCRIPTION OF THE MOTION OF A CONTINUUM

Let a fixed frame of reference $O\text{-}x_1x_2x_3$ be chosen. Let the location of a material particle be $x_1 = a_1$, $x_2 = a_2$, $x_3 = a_3$ when time $t = t_0$. We shall use (a_1, a_2, a_3) as the label for that particle. As time goes on, the particle moves. Its location has the history

$$x_1 = x_1(a_1, a_2, a_3, t), \qquad x_2 = x_2(a_1, a_2, a_3, t), \qquad x_3 = x_3(a_1, a_2, a_3, t)$$

referred to the same coordinate system or, in short,

$$x_i = x_i(a_1, a_2, a_3, t), \qquad (i = 1, 2, 3). \tag{10.2–1}$$

If such an equation is known for every particle in the body, then we know the history of motion of the entire body. Mathematically, Eq. (10.2–1) defines the *transformation*, or *mapping*, of a domain $D(a_1, a_2, a_3)$ into a domain $D'(x_1, x_2, x_3)$, with t as a parameter. An example is shown in Fig. 10.2. If the mapping is continuous and one to one—i.e., for every point (a_1, a_2, a_3), there is one and only one point (x_1, x_2, x_3) and vice versa—and neighboring points in $D(a_1, a_2, a_3)$ are mapped into neighboring points in $D'(x_1, x_2, x_3)$, then the functions $x_i(a_1, a_2, a_3, t)$ must be single valued, continuous, and continuously differentiable, and the Jacobian must not vanish in the domain D.

The mapping given by Eq. (10.2–1) is said to be a *material description* of the motion of the body. In a material description, the velocity and acceleration of the particle at (a_1, a_2, a_3) are, respectively,

$$v_i(a_1, a_2, a_3, t) = \left.\frac{\partial x_i}{\partial t}\right|_{(a_1,a_2,a_3)}, \tag{10.2–2}$$

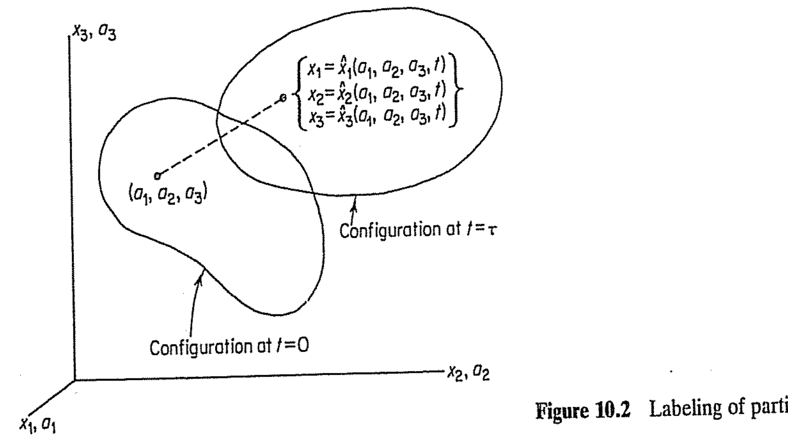

Figure 10.2 Labeling of particles.

and

$$\alpha_i(a_1, a_2, a_3, t) = \left.\frac{\partial v_i}{\partial t}\right|_{(a_1,a_2,a_3)} = \left.\frac{\partial^2 x_i}{\partial t^2}\right|_{(a_1,a_2,a_3)}. \tag{10.2–3}$$

Conservation of mass may be expressed as follows. Let $\rho(\mathbf{x})$ be the density of the material at location $\mathbf{x}$, where the symbol $\mathbf{x}$ stands for (x_1, x_2, x_3). Let $\rho_0(\mathbf{a})$ be the density at the point (a_1, a_2, a_3) when $t = 0$. Then the mass of the material enclosed in a volume V is $\int_D \rho_0(\mathbf{a})\, da_1\, da_2\, da_3$ at $t = 0$ and is $\int_{D'} \rho(\mathbf{x})\, dx_1\, dx_2\, dx_3$ at time t. Thus, conservation of mass is expressed by the formula

$$\int_{D'} \rho(\mathbf{x})\, dx_1\, dx_2\, dx_3 = \int_D \rho_0(\mathbf{a})\, da_1\, da_2\, da_3, \tag{10.2–4}$$

where the integrals extend over the same particles. But

$$\int_{D'} \rho(\mathbf{x})\, dx_1\, dx_2\, dx_3 = \int_D \rho(\mathbf{x}) \det\left|\frac{\partial x_i}{\partial a_j}\right| da_1\, da_2\, da_3, \tag{10.2–5}$$

where $|\partial x_i/\partial a_j|$ is the Jacobian of the transformation, i.e., the determinant of the matrix $(\partial x_i/\partial a_j)$:

$$\det\left|\frac{\partial x_i}{\partial a_j}\right| = \begin{vmatrix} \partial x_1/\partial a_1 & \partial x_1/\partial a_2 & \partial x_1/\partial a_3 \\ \partial x_2/\partial a_1 & \partial x_2/\partial a_2 & \partial x_2/\partial a_3 \\ \partial x_3/\partial a_1 & \partial x_3/\partial a_2 & \partial x_3/\partial a_3 \end{vmatrix}. \tag{10.2–6}$$

Identifying the right-hand sides of Eqs. (10.2–4) and (10.2–5) and realizing that the result must hold for any arbitrary domain D, we see that the integrands must be equal:

$$\rho_0(\mathbf{a}) = \rho(\mathbf{x}) \det\left|\frac{\partial x_i}{\partial a_j}\right|. \tag{10.2–7}$$

Similarly,

$$\rho(\mathbf{x}) = \rho_0(\mathbf{a}) \det \left|\frac{\partial a_i}{\partial x_j}\right|. \tag{10.2–8}$$

These equations relate the density in different configurations of the body to the transformation that leads from one configuration to another.

Thus, the material description of a continuum follows the method used in particle mechanics.

10.3 SPATIAL DESCRIPTION OF THE MOTION OF A CONTINUUM

In the material description, every particle is identified by its coordinates at a given instant of time t_0. This is not always convenient. When we describe the flow of water in a river, we do not desire to identify the location from which every particle of water comes. Instead, we are generally interested in the instantaneous velocity field and its evolution with time. This leads to the *spatial description* traditionally used in hydrodynamics. The location (x_1, x_2, x_3) and the time t are taken as independent variables. It is natural for hydrodynamics because measurements are more easily made and directly interpreted in terms of what happens at a certain place, rather than following the particles.

In a spatial description, the instantaneous motion of the continuum is described by the velocity vector field $v_i(x_1, x_2, x_3, t)$, which, of course, is the velocity of a particle instantaneously located at (x_1, x_2, x_3) at time t. We shall show that the instantaneous acceleration of the particle is given by the formula

$$\dot{v}_i(\mathbf{x}, t) = \frac{\partial v_i}{\partial t}(\mathbf{x}, t) + v_j \frac{\partial v_i}{\partial x_j}(\mathbf{x}, t), \qquad \blacktriangle \quad (10.3–1)$$

where $\mathbf{x}$ again stands for the variables x_1, x_2, x_3, and every quantity in the formula is evaluated at $(\mathbf{x}, t)$. The proof follows from the fact that a particle located at (x_1, x_2, x_3) at time t is moved to a point with coordinates $x_i + v_i\,dt$ at the time $t + dt$ and that, according to Taylor's theorem, and by omitting the higher-order infinitesimal terms as $dt \to 0$,

$$\begin{aligned}\dot{v}_i(\mathbf{x}, t)\,dt &= v_i(x_j + v_j\,dt,\, t + dt) - v_i(x_j, t)\\ &= v_i + \frac{\partial v_i(\mathbf{x}, t)}{\partial t}\,dt + \frac{\partial v_i(\mathbf{x}, t)}{\partial x_j}\,v_j\,dt - v_i,\end{aligned}$$

which reduces to Eq. (10.3–1). The first term in Eq. (10.3–1) may be interpreted as arising from the dependence of the velocity field on time, the second term as the contribution of the motion of the particle in the nonhomogeneous velocity field. Accordingly, these terms are called the *local* and the *convective* parts of the acceleration, respectively.

The reasoning that leads to Eq. (10.3–1) is applicable to any function $F(x_1, x_2, x_3, t)$ that is attributable to the moving particles, such as the temperature. A convenient terminology is the *material derivative*, which is denoted by a dot or the symbol D/Dt. Thus, the material derivative of F is

$$\dot{F} = \frac{DF}{Dt} \equiv \left(\frac{\partial F}{\partial t}\right)_{\mathbf{x}=\text{const.}} + v_1 \frac{\partial F}{\partial x_1} + v_2 \frac{\partial F}{\partial x^2} + v_3 \frac{\partial F}{\partial x_3}. \qquad \blacktriangle \quad (10.3–2)$$

On the other hand, if $F(x_1, x_2, x_3, t)$ is transformed into $F(a_1, a_2, a_3, t)$ through the transformation given by Eq. (10.2–1), then $F(a_1, a_2, a_3, t)$ is indeed the value of F attached to the particle (a_1, a_2, a_3). Hence, the material derivative F does mean the rate of change of the property F of the particle (a_1, a_2, a_3). Formally,

$$\dot{F} = \left.\frac{\partial F(a_1, a_2, a_3, t)}{\partial t}\right|_{\mathbf{a}}. \tag{10.3–3}$$

On regarding $F(x_1, x_2, x_3, t)$ as an implicit function of a_1, a_2, a_3, t, we have

$$\dot{F} = \left.\frac{\partial F}{\partial t}\right|_{\mathbf{x}} + \left.\frac{\partial F}{\partial x_1}\right|_t \left.\frac{\partial x_1}{\partial t}\right|_{\mathbf{a}} + \left.\frac{\partial F}{\partial x_2}\right|_t \left.\frac{\partial x_2}{\partial t}\right|_{\mathbf{a}} + \left.\frac{\partial F}{\partial x_3}\right|_t \left.\frac{\partial x_3}{\partial t}\right|_{\mathbf{a}}, \tag{10.3–4}$$

which reduces to Eq. (10.3–2) by virtue of Eq. (10.2–2).

10.4 THE MATERIAL DERIVATIVE OF A VOLUME INTEGRAL

Let $I(t)$ be a volume integral of a continuously differentiable function $A(\mathbf{x}, t)$ defined over a spatial domain $V(x_1, x_2, x_3, t)$ *occupied by a given set of material particles*:

$$I(t) = \iiint_V A(\mathbf{x}, t)\,dx_1\,dx_2\,dx_3. \tag{10.4–1}$$

Here again, we write $\mathbf{x}$ for x_1, x_2, x_3. The function $I(t)$ is a function of the time t because both the integrand $A(\mathbf{x}, t)$ and the domain $V(\mathbf{x}, t)$ depend on the parameter t. As t varies, $I(t)$ varies also, and we ask: What is the rate of change of $I(t)$ with respect to t? This rate, denoted by DI/Dt and called the *material derivative* of I, is defined for a given set of material particles.

The phrase "for a given set of particles" is of primary importance. The question is how fast the material body itself "sees" the value of I changing. To evaluate this rate, note that the boundary S of the body at the instant t will have moved at time $t + dt$ to a neighboring surface S', which bounds the domain V' (Fig. 10.3). The material derivative of I is defined as

$$\frac{DI}{Dt} = \lim_{dt \to 0} \frac{1}{dt}\left[\int_{V'} A(\mathbf{x}, t + dt)\,dV - \int_V A(\mathbf{x}, t)\,dV\right]. \tag{10.4–2}$$

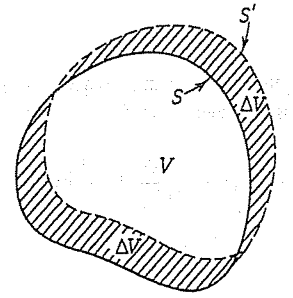

Figure 10.3 Continuous change of the boundary of a region.

Attention is drawn to the difference in the domains V' and V. Let ΔV be the domain $V' - V$. We note that ΔV is swept out by the motion of the surface S in the small time interval dt. Since $V' = V + \Delta V$, we can write Eq. (10.4–2) as

$$\begin{aligned}\frac{DI}{Dt} &= \lim_{dt\to 0}\frac{1}{dt}\left[\int_V A(\mathbf{x}, t+dt)\,dV + \int_{\Delta V} A(\mathbf{x}, t+dt)\,dV \right.\\ &\qquad \left. - \int_V A(\mathbf{x}, t)\,dV\right] \\ &= \lim_{dt\to 0}\left\{\frac{1}{dt}\int_V [A(\mathbf{x}, t+dt) - A(\mathbf{x}, t)]\,dV \right.\\ &\qquad \left. + \frac{1}{dt}\int_{\Delta V} A(\mathbf{x}, t+dt)\,dV\right\}.\end{aligned} \qquad (10.4\text{–}3)$$

For a continuously differentiable function $A(\mathbf{x}, t)$, the first term on the right-hand side contributes the value $\int_V \partial A/\partial t\, dV$ to DI/Dt. The last term may be evaluated by noting that for an infinitesimal dt, the integrand may be taken to be $A(\mathbf{x}, t)$ on the boundary surface S [because of the assumed continuity of $A(\mathbf{x}, t)$] and that the integral is equal to the sum of $A(\mathbf{x}, t)$ multiplied by the volume swept out by the particles situated on the boundary S in the time interval dt. If n_i is the unit vector along the outer normal of S, then, since the displacement of a particle on the boundary is $v_i\, dt$, the volume swept out by particles occupying an element of area dS on the boundary S is $dV = n_i v_i dS \cdot dt$. On ignoring infinitesimal quantities of the second or higher order, we see the contribution of this element to DI/Dt is Av_in_idS. The total contribution is obtained by an integration over S. Therefore,

$$\frac{D}{Dt}\int_V A\,dV = \int_V \frac{\partial A}{\partial t}\,dV + \int_S Av_jn_j dS. \qquad ▲\ (10.4\text{–}4)$$

Transforming the last integral by Gauss's theorem and using Eq. (10.3–2), we have

$$\begin{aligned}\frac{D}{Dt}\int_V A\,dV &= \int_V \frac{\partial A}{\partial t}\,dV + \int_V \frac{\partial}{\partial x_j}(Av_j)\,dV \\ &= \int_V \left(\frac{\partial A}{\partial t} + v_j\frac{\partial A}{\partial x_j} + A\frac{\partial v_j}{\partial x_j}\right) dV \qquad ▲\ (10.4\text{–}5)\\ &= \int_V \left(\frac{DA}{Dt} + A\frac{\partial v_j}{\partial x_j}\right) dV.\end{aligned}$$

This important formula will be used repeatedly in the sections that follow. It should be noted that according to Eq. (10.4–5), *the operation of forming the material derivative and that of spatial integration are noncommutative in general.*

10.5 THE EQUATION OF CONTINUITY

The law of conservation of mass was discussed in Sec. 10.2. With the results of Sec. 10.4, we can now give some alternative forms.

The mass contained in a domain V at a time t is

$$m = \int_V \rho\,dV, \qquad (10.5\text{–}1)$$

where $\rho = \rho(\mathbf{x}, t)$ is the density of the continuum at location $\mathbf{x}$ at time t. Conservation of mass requires that $Dm/Dt = 0$. The derivative Dm/Dt is given by Eq. (10.4–4) or Eq. (10.4–5) if A is replaced by ρ. Since the result must hold for an arbitrary domain V, the integrand must vanish. Hence, we obtain the following forms of the law of conservation of mass enclosed in a surface S with outer normal $\mathbf{n}$:

$$\int_V \frac{\partial \rho}{\partial t}\,dV + \int_S \rho v_j n_j\,dS = 0. \qquad ▲\ (10.5\text{–}2)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial\, \rho v_j}{\partial x_j} = 0. \qquad ▲\ (10.5\text{–}3)$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_j}{\partial x_j} = 0. \qquad ▲\ (10.5\text{–}4)$$

These are called the *equations of continuity*. The integral form, Eq. (10.5–2), is useful when the differentiability of ρv_j cannot be assumed.

In problems of statics, these equations are satisfied identically. Then the conservation of mass must be expressed by Eq. (10.2–7) or Eq. (10.2–8).

10.6 THE EQUATIONS OF MOTION

Newton's laws of motion state that in an inertial frame of reference, the material rate of change of the linear momentum of a body is equal to the resultant of the forces applied to the body.

At an instant of time t, the linear momentum of all the particles contained in a domain V is

$$\mathcal{P}_i = \int_V \rho v_i \, dV. \qquad (10.6\text{–}1)$$

If the body is subjected to surface tractions $\overset{\nu}{T}_i$ and a body force per unit volume X_i, the resultant force is

$$\mathcal{F}_i = \int_S \overset{\nu}{T}_i \, dS + \int_V X_i \, dV. \qquad (10.6\text{–}2)$$

According to Cauchy's formula, Eq. (3.3–2), the surface traction may be expressed in terms of the stress field σ_{ij}, so that $\overset{\nu}{T}_i = \sigma_{ji}\nu_j$, where ν_j is the unit vector along the outer normal to the boundary surface S of the domain V. On substituting $\sigma_{ij}\nu_j$ for $\overset{\nu}{T}_i$ into Eq. (10.6–2) and transforming the surface integral into a volume integral by Gauss's theorem, we have

$$\mathcal{F}_i = \int_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + X_i\right) dV. \qquad (10.6\text{–}3)$$

Newton's law states that

$$\frac{D}{Dt}\mathcal{P}_i = \mathcal{F}_i. \qquad (10.6\text{–}4)$$

Hence, according to Eq. (10.4–5), with A identified with ρv_i, we have

$$\int_V \left[\frac{\partial\, \rho v_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j)\right] dV = \int_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + X_i\right) dV. \qquad (10.6\text{–}5)$$

Since this equation must hold for an arbitrary domain V, the integrands on the two sides must be equal. Thus,

$$\frac{\partial\, \rho v_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \frac{\partial \sigma_{ij}}{\partial x_j} + X_i. \qquad (10.6\text{–}6)$$

The left-hand side of Eq. (10.6–6) is equal to

$$v_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial\, \rho v_j}{\partial x_j}\right) + \rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right).$$

The quantity in the first set of parentheses vanishes according to the equation of continuity, Eq. (10.5–3), while that in the second set of parentheses is the acceleration Dv_i/Dt. Hence, we obtain the celebrated *Eulerian equation of motion of a continuum*:

$$\rho \frac{Dv_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + X_i. \qquad ▲\ (10.6\text{–}7)$$

The equation of equilibrium discussed in Sec. 3.4 is a special case that can be obtained by setting all velocity components v_i equal to zero.

10.7 MOMENT OF MOMENTUM

An application of the law of balance of angular momentum to the particular case of *static equilibrium* leads to the conclusion that stress tensors are symmetric tensors (see Sec. 3.4.) We shall now show that no additional restriction to the motion of a continuum is introduced in dynamics by the angular momentum postulate, which states that the material rate of change of the moment of momentum with respect to an origin is equal to the resultant moment of all the applied forces about the same origin.

At an instant of time t, a body occupying a regular region V of space with boundary S has the moment of momentum [See Eq. (3.2–2)]

$$\mathcal{H}_i = \int_V e_{ijk} x_j \rho v_k \, dV \qquad (10.7\text{–}1)$$

with respect to the origin of coordinates. If the body is subjected to surface traction $\overset{\nu}{T}_i$ and a body force per unit volume X_i, the resultant moment about the origin is

$$\mathcal{L}_i = \int_V e_{ijk} x_j X_k \, dV + \int_S e_{ijk} x_j \overset{\nu}{T}_k \, dS. \qquad (10.7\text{–}2)$$

Introducing Cauchy's formula, $\overset{\nu}{T}_k = \sigma_{lk} n_l$, into the last integral, and transforming the result into a volume integral by Gauss's theorem, we obtain

$$\mathcal{L}_i = \int_V e_{ijk} x_j X_k \, dV + \int_V (e_{ijk} x_j \sigma_{lk})_{,l} \, dV. \qquad (10.7\text{–}3)$$

Euler's law states that, for any region V,

$$\frac{D}{Dt}\mathcal{H}_i = \mathcal{L}_i. \qquad (10.7\text{–}4)$$

Evaluating the material derivative of $\mathcal{H}_i$ according to Eq. (10.4–5) and using Eq. (10.7–3), we obtain

$$e_{ijk} x_j \frac{\partial}{\partial t}(\rho v_k) + \frac{\partial}{\partial x_l}(e_{ijk} x_j \rho v_k v_l) = e_{ijk} x_j X_k + e_{ijk}(x_j \sigma_{lk})_{,l}. \qquad (10.7\text{–}5)$$

The second term in Eq. (10.7–5) can be written as

$$e_{ijk}\rho v_j v_k + e_{ijk}x_j \frac{\partial}{\partial x_l}(\rho v_k v_l) = 0 + e_{ijk}x_j \frac{\partial}{\partial x_l}(\rho v_k v_l)$$

because e_{ijk} is antisymmetric and $v_j v_k$ is symmetric with respect to j, k. The last term in Eq. (10.7–5) can be written as $e_{ijk}\sigma_{jk} + e_{ijk}x_j\sigma_{lk,l}$. Hence, Eq. (10.7–5) becomes

$$e_{ijk}x_j\left[\frac{\partial}{\partial t}(\rho v_k) + \frac{\partial}{\partial x_l}(\rho v_k v_l) - X_k - \sigma_{lk,l}\right] - e_{ijk}\sigma_{jk} = 0. \qquad (10.7\text{–}6)$$

By the equation of motion Eq. (10.6–6), the sum in the square brackets vanishes. Hence, Eq. (10.7–6) is reduced to

$$e_{ijk}\sigma_{jk} = 0; \qquad (10.7\text{–}7)$$

i.e., $\sigma_{jk} = \sigma_{kj}$. Thus, if the stress tensor is symmetric, the law of balance of moment of momentum is satisfied identically.

10.8 THE BALANCE OF ENERGY

The motion of a continuum must be governed further by the law of conservation of energy. If mechanical energy alone is of interest in a problem, then the energy equation is merely the first integral of the equation of motion. If a thermal process is significant, then the equation of energy becomes an independent equation to be satisfied.

The law of conservation of energy is the first law of thermodynamics. Its expression for a continuum can be derived as soon as all forms of energy and work are listed. Let us consider a continuum for which there are three forms of energy: the kinetic energy K, the gravitational energy G, and the internal energy E. We have

$$\text{Energy} = K + G + E. \qquad (10.8\text{–}1)$$

The *kinetic energy* contained in a regular domain V at a time t is

$$K = \int \frac{1}{2}\rho v_i v_i \, dV, \qquad (10.8\text{–}2)$$

where v_i are the components of the velocity vector of a particle occupying an element of volume dV and ρ is the density of the material. The *gravitational* energy depends on the distribution of mass and may be written as

$$G = \int \rho\phi(x)\, dV, \qquad (10.8\text{–}3)$$

where ϕ is the gravitational potential per unit mass. In the important special case of a uniform gravitational field, we have

$$G = \int \rho g z \, dV, \qquad (10.8\text{–}4)$$

where g is the gravitational acceleration and z is a distance measured from a certain plane in a direction opposite to the gravitational field. The *internal energy* is written in the form

$$E = \int \rho E \, dV, \qquad (10.8\text{–}5)$$

where E is the *internal energy per unit mass*. The first law of thermodynamics states that the energy of a system can be changed by absorption of heat Q and by work W done on the system:

$$\Delta \text{ energy} = Q + W. \qquad (10.8\text{–}6)$$

Expressing this in terms of rates, we have

$$\frac{D}{Dt}(K + G + E) = \dot{Q} + \dot{W}, \qquad (10.8\text{–}7)$$

where $\dot{Q}$ and $\dot{W}$ are the rates of change of Q and W per unit time.

Now, the heat input into the body must be imparted through the boundary. To describe the heat flow, a *heat flux* vector $\mathbf{h}$ (with components h_1, h_2, h_3) is defined as follows. Let dS be a surface element in the body, with unit outer normal n_i. Then the rate at which heat is transmitted across the surface dS in the direction of v_i is assumed to be representable as $h_i n_i \, dS$. If the medium is moving, we insist that the surface element dS be composed of the same particles. The rate of heat input is, therefore,

$$\dot{Q} = -\int_S h_i n_i \, dS = -\int_V \frac{\partial h_i}{\partial x_i} dV. \qquad (10.8\text{–}8)$$

The rate at which work is done on the body by the body force per unit volume F_i in V and the surface tractions $\overset{\nu}{T}_i$ in S is the *power*

$$\begin{aligned}\dot{W} &= \int F_i v_i \, dV + \int \overset{\nu}{T}_i n_i \, dS \\ &= \int F_i v_i \, dV + \int \sigma_{ij} n_j n_i \, dS \qquad (10.8\text{–}9)\\ &= \int F_i v_i \, dV + \int (\sigma_{ij} n_i)_{,j} \, dV.\end{aligned}$$

Since, in Eq. (10.8–7), the gravitational energy is included in the term G, the power W must be evaluated with the gravitational force excluded from the body force F_i. Substituting Eqs. (10.8–2), (10.8–3), (10.8–5), (10.8–8), and (10.8–9) into the first law of thermodynamics, Eq. (10.8–7), and using Eq. (10.4–5) to compute the material derivatives, we obtain the following result after some calculation:

$$\frac{1}{2}\rho\frac{Dv^2}{Dt} + \frac{v^2}{2}\frac{D\rho}{Dt} + \frac{v^2}{2}\rho\,\mathrm{div}\,\mathbf{v} + \rho\frac{DE}{Dt} + E\frac{D\rho}{Dt} \tag{10.8–10}$$

$$+ E\rho\,\mathrm{div}\,\mathbf{v} + \rho\frac{D\phi}{Dt} + \phi\frac{D\rho}{Dt} + \phi\rho\,\mathrm{div}\,\mathbf{v}$$

$$= -\frac{\partial h_i}{\partial x_i} + F_i v_i + \sigma_{ij,j}v_i + \sigma_{ij}v_{i,j}.$$

This equation can be simplified greatly if we make use of the equations of continuity and motion:

$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\mathbf{v} = 0, \qquad \rho\frac{Dv_i}{Dt} = X_i + \sigma_{ij,j}. \tag{10.8–11}$$

Here, X_i is the total body force per unit mass. The difference between X_i and F_i is the gravitational force and is, by definition,

$$X_i - F_i = -\rho\frac{\partial\phi}{\partial x_i}. \tag{10.8–12}$$

Since

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + v_i\frac{\partial\phi}{\partial x_i},$$

and $\partial\phi/\partial t = 0$ for a gravitational field that is independent of time, we have, for such a field, and with Eqs. (10.8–11) and (10.8–12),

$$\frac{1}{2}\rho\frac{Dv^2}{Dt} + \rho\frac{DE}{Dt} = -\frac{\partial h_i}{\partial x_i} + \rho v_i\frac{Dv_i}{Dt} + \sigma_{ij}v_{i,j}. \tag{10.8–13}$$

But

$$\rho v_i\frac{Dv_i}{Dt} = \frac{1}{2}\rho\frac{Dv^2}{Dt}, \tag{10.8–14}$$

and

$$\sigma_{ij}v_{i,j} = \sigma_{ij}[\tfrac{1}{2}(v_{i,j} + v_{j,i}) + \tfrac{1}{2}(v_{i,j} - v_{j,i})] = \sigma_{ij}V_{ij} + 0, \tag{10.8–15}$$

where

$$V_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \tag{10.8–16}$$

is the *strain-rate tensor*. The last term in Eq. (10.8–15) vanishes because it is the contraction of the product of a symmetric tensor σ_{ij} with an antisymmetric one. Hence, Eq. (10.8–13) can be simplified, and we obtain the final form of the energy equation:

$$\rho\frac{DE}{Dt} = -\frac{\partial h_i}{\partial x_i} + \sigma_{ij}V_{ij}. \tag{10.8–17}$$

Specialization

(A) If all the nonmechanical transfer of energy consists of heat conduction, which obeys Fourier's law,

$$h_i = -J\lambda\frac{\partial T}{\partial x_i}, \tag{10.8–18}$$

where J is the mechanical equivalent of heat, λ is the conductivity, and T is the absolute temperature, then the energy equation becomes

$$\rho\frac{DE}{Dt} = J\frac{\partial}{\partial x_i}\left(\lambda\frac{\partial T}{\partial x_i}\right) + \sigma_{ij}V_{ij}. \tag{10.8–19}$$

(B) The usual equation of heat conduction in a continuum at rest is obtained by deleting the terms involving ϕ, v_i, and V_{ij} and setting

$$E = JcT, \tag{10.8–20}$$

where c is the specific heat for the vanishing rate of deformation. Then Eq. (10.8–19) becomes

$$\rho c\frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda\frac{\partial T}{\partial x_i}\right). \tag{10.8–21}$$

10.9 THE EQUATIONS OF MOTION AND CONTINUITY IN POLAR COORDINATES

In Secs. 3.6 and 5.8, we considered the stress and strain components, respectively, in polar coordinates. The corresponding equations of motion and continuity can be derived in the same manner: by the method of general tensor analysis, by transformation from the Cartesian coordinates, or by direct ad hoc derivation from first principles. Illustrations of the last two approaches follow.

The basic equations for transformation between Cartesian coordinates x, y, z and polar coordinates r, θ, z are given in Sec. 5.8. If we substitute Eq. (3.6–5) into the equation of equilibrium,

$$\frac{\partial\sigma_{ij}}{\partial x_j} = 0, \tag{10.9–1}$$

i.e.,

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0,$$

etc., and use Eq. (5.8–3) to transform the derivatives, we obtain

$$\left(\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{\partial\sigma_{rz}}{\partial z}\right)\cos\theta$$
$$-\left(\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{r\theta}}{\partial r}+2\frac{\sigma_{\theta}}{r}+\frac{\partial\sigma_{\theta z}}{\partial z}\right)\sin\theta=0. \qquad (10.9\text{–}2)$$

Since this equation must hold for all values of θ, we must have, at $\theta = 0$ and at $\theta = \pi/2$, respectively,

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{\partial\sigma_{rz}}{\partial z}=0,$$
$$\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{r}+\frac{\partial\sigma_{z\theta}}{\partial z}=0. \qquad (10.9\text{–}3)$$

But the choice of the x-direction is arbitrary, so Eq. (10.9–3) must be valid for all values of θ. Similarly, from Eq. (10.9–1) with $i = 3$, we obtain the third equation of equilibrium,

$$\frac{\partial\sigma_{zz}}{\partial z}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zr}}{\partial r}+\frac{\sigma_{rz}}{r}=0. \qquad (10.9\text{–}4)$$

If the continuum is subjected to an acceleration and a body force, then the equation of motion, Eq. (10.6–7), is

$$\frac{\partial\sigma_{ij}}{\partial x_j}+X_i=\rho\frac{Dv_i}{Dt}=\rho\, a_i. \qquad (10.9\text{–}5)$$

The body force per unit volume may be resolved into components F_r, F_θ, F_z along the r-, θ-, and z-directions, respectively. The acceleration $Dv_i/Dt = a_i$ must be considered carefully. The component of acceleration in the x-direction in rectangular coordinates is

$$a_x=\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}. \qquad (10.9\text{–}6)$$

The components of acceleration a_x, a_y, a_z and of velocity v_x, v_y, v_z are related to the components a_r, a_θ, a_z and v_r, v_θ, v_z in polar coordinates by the same Eqs. (5.8–4) that relate the displacements, provided that u is replaced by a and v, respectively. Hence, by substitution of Eqs. (5.8–3) and (5.8–4) into Eq. (10.9–6), we obtain

$$a_x=\frac{\partial}{\partial t}(v_r\cos\theta-v_\theta\sin\theta)$$
$$+(v_r\cos\theta+v_\theta\sin\theta)\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(v_r\cos\theta-v_\theta\sin\theta)$$

$$+(v_r\sin\theta+v_\theta\cos\theta)\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(v_r\cos\theta-v_\theta\sin\theta)$$
$$+v_z\frac{\partial}{\partial z}(v_r\cos\theta-v_\theta\sin\theta)$$
$$=\cos\theta\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$
$$-\sin\theta\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right). \qquad (10.9\text{–}7)$$

Comparing Eq. (10.9–7) with the equation

$$a_x=a_r\cos\theta-a_\theta\sin\theta, \qquad (10.9\text{–}8)$$

we obtain the components of acceleration:

$$a_r=\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z},$$
$$a_\theta=\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}. \qquad (10.9\text{–}9)$$

Similarly,

$$a_z=\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}. \qquad (10.9\text{–}10)$$

The full equations of motion are

$$\rho a_r=\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{\partial\sigma_{rz}}{\partial z}+F_r,$$
$$\rho a_\theta=\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{\partial r}+\frac{\partial\sigma_{r\theta}}{\partial z}+F_\theta, \qquad (10.9\text{–}11)$$
$$\rho a_z=\frac{\partial\sigma_{zz}}{\partial z}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zr}}{\partial r}+\frac{\sigma_{rz}}{r}+F_z.$$

These derivations are again straightforward, but not very instructive from the physical point of view. A second derivation based on an examination of the balance of forces acting on an element may supply further insight into the equations. Figure 10.4 shows the free-body diagram for an isolated element with the stress pattern

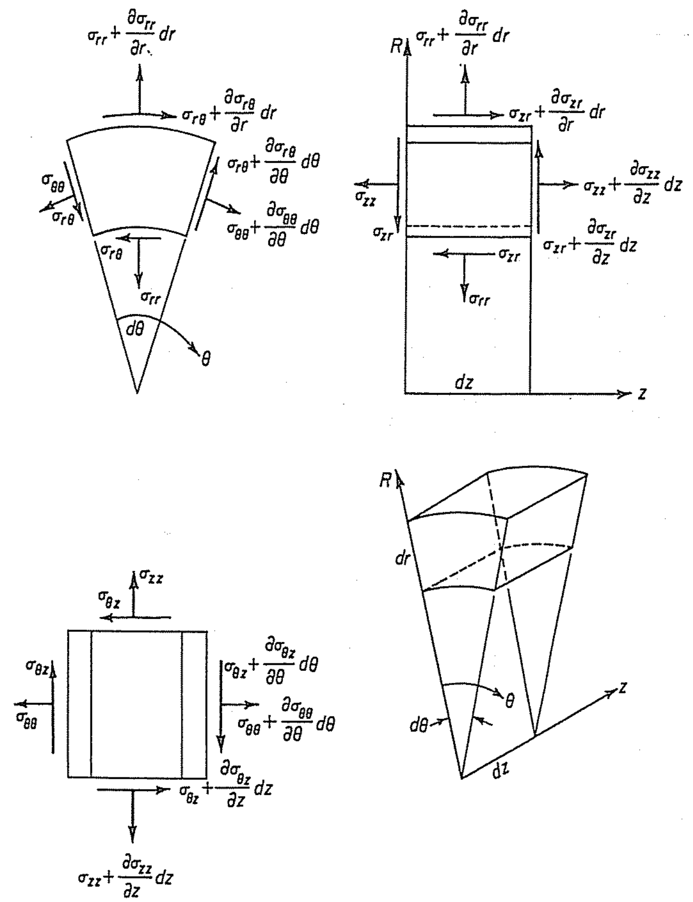

Figure 10.4 Stress field in cylindrical polar coordinates.

indicated. The equation of motion indicates that the acceleration in the radial direction is equal to the sum of all the forces acting in the radial direction. Thus,

$$\rho a_r\, dr\, dz\left[\frac{r\, d\theta + (r + dr)d\theta}{2}\right] = F_r\, dr\, dz\left[\frac{r\, d\theta + (r + dr)d\theta}{2}\right]$$

$$+ \left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r} dr\right)(r + dr)d\theta\, dz - \sigma_{rr} r\, d\theta\, dz$$

$$- \sigma_{\theta\theta}\, dr\, dz \sin\frac{d\theta}{2} - \left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} d\theta\right) dr\, dz \sin\frac{d\theta}{2}$$

$$+ \left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\right) dr\, dz - \sigma_{r\theta}\, dr\, dz$$

$$+ \left(\sigma_{rz} + \frac{\partial \sigma_{rz}}{\partial z} dz - \sigma_{rz}\right)\left[\frac{r\, d\theta + (r + dr)\, d\theta}{2}\right] dr. \qquad (10.9\text{–}12)$$

Expanding, dropping higher-order infinitesimal quantities, and dividing through by r, we obtain the first equation of Eq. (10.9–11). The other equations can be obtained in a similar manner. Note that in the equation for radial equilibrium, the term $-\sigma_{\theta\theta}/r$ is a radial pressure in the nature of hoop stress; the term σ_{rr}/r is the contribution due to the larger area of the outer surface at $r + dr$ than that at radius r. The term σ_{rz}/r in the equation for axial equilibrium is present for the same reason. The term $2\sigma_{r\theta}/r$ in the tangential equation has two origins: One is for the same reason as before, viz., that the outer surface is larger; the other arises from the fact that the radial surfaces at θ and $\theta + d\theta$ are not parallel, but make an angle $d\theta$.

A similar graphical interpretation can be made of the individual terms in the expressions for acceleration. The term $-v_\theta^2/r$ in a_r is of the nature of centripetal acceleration. The term $v_\theta v_r/r$ in a_θ arises from the rotation of the radial velocity vector v_r, thus contributing a tangential component of acceleration.

A similar treatment can be used to transform the equation of continuity, Eq. (10.5–3), into polar coordinates. But here it is perhaps most instructive to study the balance of mass flow in an element, as shown in Fig. 10.5. With the area through which the mass flow takes place accounted for properly, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial \rho v_\theta}{\partial \theta} + \frac{\partial \rho v_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0. \qquad (10.9\text{–}13)$$

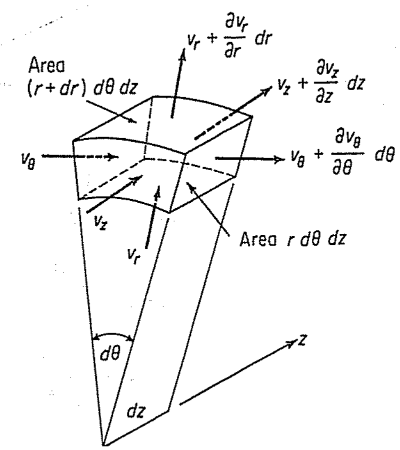

Figure 10.5 Conservation of mass in cylindrical polar coordinates.

PROBLEMS

10.1 State the definitions of (a) a line integral, (b) a surface integral, and (c) a volume integral.

10.2 State the mathematical conditions under which Eqs. (10.1–4), (10.1–5), (10.4–4), and (10.4–5) are valid.

10.3 Evaluate the line integral

$$\oint_C y^2\,dx + x^2\,dy,$$

where C is a triangle with vertices (1,0), (1,1), (0,0). (See Fig. P10.3.)

Answer: 1/3.

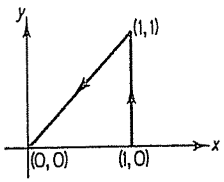

Figure P10.3 Path of integration.

10.4 Evaluate $\oint_C (x^2 - y^2)\,ds$, where C is the circle $x^2 + y^2 = 4$.

10.5 Derive Green's theorem: Let D be a domain of the xy-plane, and let C be a piecewise smooth simple closed curve in D whose interior is also in D. Let $P(x, y)$ and $Q(x, y)$ be functions that are defined and continuous in D and that have continuous first partial derivatives in D. Then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

where R is the closed region bounded by C.

10.6 Interpret Green's theorem vectorially to derive the following theorems:

$$\text{(a)}\ \oint_C u_T\,ds = \iint_R \text{curl}_z\,\mathbf{u}\,dx\,dy,$$

$$\text{(b)}\ \oint_C v_n\,ds = \iint_R \text{div}\,\mathbf{v}\,dx\,dy,$$

where $\mathbf{u}$, $\mathbf{v}$ are vector fields, u_T is the tangential component of $\mathbf{u}$ (tangent to the curve C), ds is the arc length, and v_n is the normal component of $\mathbf{v}$ on C. Equation (a) is a special case of Stokes's theorem. Equation (b) is the two-dimensional form of Gauss's theorem.

10.7 A rubber spherical balloon is quickly blown up in an angry sea by a ditched pilot. Let a particle on the balloon be located at

$$x = x(t), \qquad y = y(t), \qquad z = z(t).$$

Let the surface of the balloon be described by the equation

$$F(t) = (x - \lambda)^2 + (y - \mu)^2 + (z - \nu)^2 - a^2 = 0,$$

where $\lambda(t)$, $\mu(t)$, and $\nu(t)$, which define the center of the sphere, and $a(t)$, the radius, are functions of time. (See Fig. P10.7.)

Show that $DF/Dt = 0$.

Derive the boundary conditions for the air and water moving about the balloon.

Solution: The equation $F(t) = 0$ representing the surface of the balloon is true at all times. Therefore, its derivative with respect to t must vanish. Since x, y, z are coordinates of the particles, and $F(t)$ is associated with the balloon at all times, the time derivative is the material derivative, i.e., DF/Dt, which is zero.

Conversely, from the equation $DF/Dt = 0$, we conclude that F = const. for a given set of particles. In particular, if the set of particles is defined by the equation $F = 0$, it remains the same set. If $F = 0$ defines the balloon at $t = 0$, it defines the balloon at any t.

The equation becomes more significant if we consider the fluid (air and water) around the balloon. Fluid particles once in contact with the balloon remain in contact with it (the so-called no-slip condition of a viscous fluid in contact with a solid body). Hence, the boundary conditions of the flow field are $F = 0$ and $DF/Dt = 0$.

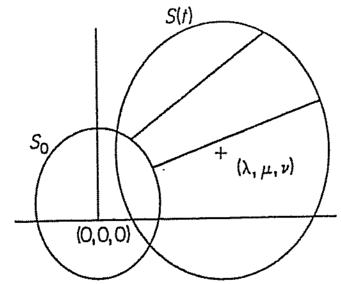

Figure P10.7 Expanding balloon.

10.8 The surface of a flag fluttering in the wind is described by the equation

$$F(x, y, z, t) = 0.$$

Write down analytically the constraints imposed by the flag on the airstream. In other words, given the shape of the boundary surface $F = 0$, derive the boundary condition for the flow. For this problem, consider the air a nonviscous fluid.

What difference would it make if the air were taken to be a viscous fluid?

Solution: As in Prob. 10.7, the boundary condition of the airstream on the flag surface $F = 0$ is

$$\frac{\partial F}{\partial t} + u_x\frac{\partial F}{\partial x} + u_y\frac{\partial F}{\partial y} + u_z\frac{\partial F}{\partial z} = 0 \tag{1}$$

where $\mathbf{u}$ (u_x, u_y, u_z) is the velocity vector. For the surface $F(x, y, z, t) = 0$, the vector $\mathbf{n}$ with components

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$$

is normal to the surface. Hence, Eq. (1) may be written

$$\frac{\partial F}{\partial t} + \mathbf{u}\cdot\mathbf{n} = 0. \tag{2}$$

This means that the normal velocity must be equal to $-\partial F/\partial t$ on the flag surface.

For a viscous fluid, the *no-slip* condition requires, in addition, that $F = 0$. (See the discussion in Sec. 11.2, p. 233.)

10.9 Two components of the velocity field of a fluid are known in the region $-2 \le x, y, z \le 2$:

$$u = (1 - y^2)(a + bx + cx^2), \qquad w = 0.$$

The fluid is incompressible. What is the velocity component v in the direction of the y-axis?

10.10 Let the temperature field of the fluid described in Prob. 10.9 be

$$T = T_0 e^{-kt} \sin \alpha x \cos \beta y.$$

Find the material rate of change of the temperature of a particle located at the origin $x = y = z = 0$. Find the same for a particle at $x = y = z = 1$.

10.11 For an isotropic Newtonian viscous fluid, derive an equation of motion expressed in terms of the velocity components.

10.12 The entropy of a moving continuum is $s(x_1, x_2, x_3, t)$ *per unit mass* of the medium. The mass density of the medium is $\rho(x_1, x_2, x_3, t)$. The velocity field is $v_i(x_1, x_2, x_3, t)$. Consider the total amount of entropy in a certain volume of the medium at a certain time. Express the rate of change of the total entropy of the material enclosed in this volume in the form of a volume integral.

11 FIELD EQUATIONS AND BOUNDARY CONDITIONS IN FLUID MECHANICS

We have acquired enough basic equations to deal with a broad range of problems. Most objects on a scale that we can see are continua. Their motion follows the laws of conservation of mass, momentum, and energy. With the proper constitutive equations and boundary conditions, we can describe many physical problems mathematically. In this chapter, we illustrate the formulation of some problems on the flow of fluids.

11.1 THE NAVIER-STOKES EQUATIONS

Let us derive the basic equations governing the flow of a Newtonian viscous fluid. Let x_1, x_2, x_3 or x, y, z be rectangular Cartesian coordinates. Let the velocity components along the x-, y-, z-axis directions be denoted by v_1, v_2, v_3 or u, v, w, respectively. Let p denote pressure; σ_{ij} or σ_{xx}, σ_{xy}, etc., be the stress components; and μ be the coefficient of viscosity. Here, and hereinafter, all Latin indices range over 1, 2, 3. Then, the stress–strain-rate relationship is given by Eq. (7.3–3):

$$\sigma_{ij} = -p\delta_{ij} + \lambda V_{kk}\delta_{ij} + 2\mu V_{ij} = -p\delta_{ij} + \lambda \frac{\partial v_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right); \tag{11.1–1}$$

i.e.,

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),$$
$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),$$
$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \tag{11.1–1a}$$
$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad \sigma_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$
$$\sigma_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).$$

Substituting these into the equation of motion, Eq. (10.6–7), we obtain the Navier-Stokes equations,

$$\rho \frac{Dv_i}{Dt} = \rho X_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial v_k}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left(\mu \frac{\partial v_k}{\partial x_i}\right) + \frac{\partial}{\partial x_k}\left(\mu \frac{\partial v_i}{\partial x_k}\right), \tag{11.1–2}$$

where X_i stands for the body force per unit mass.

The velocity components must satisfy the equation of continuity, Eq. (10.5–3), derived from the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_k)}{\partial x_k} = 0. \tag{11.1–3}$$

These equations are to be supplemented by the equations of thermal state, balance of energy, and heat flow.

If the fluid is *incompressible*, then

$$\rho = \text{const.}, \tag{11.1–4}$$

and no thermodynamic considerations need be introduced explicitly. Limiting ourselves to an incompressible homogeneous fluid, we see that the equation of continuity becomes

$$\frac{\partial v_k}{\partial x_k} = 0, \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{11.1–5}$$

and the Navier-Stokes equation is simplified to

$$\rho \frac{Dv_i}{Dt} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_k \partial x_k}. \tag{11.1–6}$$

Written out *in extenso*, these are

$$\begin{aligned} \frac{Du}{Dt} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u, \\ \frac{Dv}{Dt} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v, \\ \frac{Dw}{Dt} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w, \end{aligned} \tag{11.1–7}$$

where $\nu = \mu/\rho$ is the *kinematic viscosity* and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{11.1–8}$$

is the *Laplacian operator*. Equations (11.1–5) and (11.1–7) comprise four equations for the four variables u, v, w, and p occurring in an incompressible viscous flow.

The solution of the Navier-Stokes equation is the central problem in fluid mechanics. This equation embraces a tremendous range of physical phenomena

and has many applications to science and engineering. The equation is nonlinear and is, in general, very difficult to solve.

To complete the formulation of a problem, we must specify the boundary conditions. In Sec. 11.2, we consider the no-slip condition on a solid-fluid interface. In Sec. 11.3, the condition at a "free," or fluid-fluid, interface is considered, where surface tension plays an important role. Then a dimensional analysis is presented to illustrate the significance of the Reynolds number. We shall then consider the laminar flow in a channel or a tube as an example of a simplified solution when the nonlinear terms can be ignored. As a warning that turbulences may intervene, we discuss the classical experiments of Reynolds in Sec. 11.5.

In some instances, the viscosity of a fluid may be ignored completely, and we deal with the idealized world of "perfect fluids." In association with this idealization, the boundary conditions must be changed: The order of the differential equation would be too low to permit the satisfaction of all the boundary conditions of a viscous fluid. We relinquish the no-slip condition at the solid-fluid interface and ignore any shear gradient requirement at a free surface. As a consequence, sometimes the resulting simpler mathematical problems lead to difficulties in physical interpretations.

11.2 BOUNDARY CONDITIONS AT A SOLID-FLUID INTERFACE

One of the boundary conditions that must be satisfied at a solid-fluid interface is that the fluid must not penetrate the solid if it is impermeable to the fluid. Most containers of fluids are of this nature. Mathematically, this requires that the relative velocity component of the fluid *normal* to the solid surface must vanish.

The specification of the tangential component of velocity of the fluid relative to the solid requires much greater care. It is customary to assume that the *no-slip condition* prevails at an interface between a viscous fluid and a solid boundary. In other words, on the solid-fluid boundary, the velocities of the fluid and the solid are exactly equal. This conviction was realized only after a long historical development by comparing theoretical and experimental results.

If the solid boundary is stationary, the no-slip condition requires that the velocity change continuously from zero at the surface to the free-stream value some distance away. This boundary condition is in drastic contrast to that which is required of a nonviscous fluid, for which we can specify only that no fluid shall penetrate the solid surface; but the fluid must be permitted to slide over the solid so that their tangential velocities can be different. This is a penalty for the idealization of complete absence of viscosity. Figure 11.1 illustrates the difference. In Fig. 11.1(a), the flow of a nonviscous fluid over a stationary solid object is shown. At the interface, the fluid slips over the solid with a tangential velocity. In Fig. 11.1(b), it is shown that for a viscous fluid, the velocity must vanish on the interface.

Since the no-slip condition must be imposed for all real fluid, no matter how small the viscosity, the illustration in Fig. 11.1(b) must prevail for all real fluids.

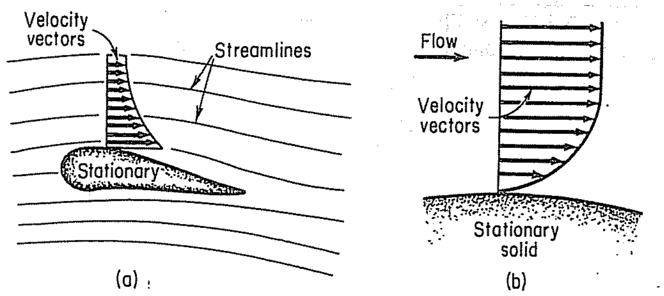

Figure 11.1 The difference in boundary conditions for flows of ideal and real fluids over a solid body. (a) Ideal fluid; (b) Real fluid.

It is known from wind-tunnel measurements that the flow field is well represented by Fig. 11.1(a) for the airfoil shown; i.e., except for the immediate neighborhood of the solid boundary, the flow can be obtained as though air had no viscosity. Yet we know that air has viscosity, even though very small. Therefore, the no-slip condition must prevail. How can we resolve this conflict?

The answer to this question and the resolution of the conflict are a triumph of modern fluid mechanics. The modern view is that the illustration shown in Fig. 11.1(b) is an indication of what happens in the immediate neighborhood of a solid boundary. We should consider that figure as an enlargement of what happens in a very small region of a flow next to an interface. This region is the *boundary layer*. Beyond the boundary layer, the flow is practically nonviscous. The dramatic importance of the boundary layer will be seen at the sharp trailing edge of the airfoil. It dictates the condition that the flow must leave the sharp trailing edge smoothly, with no discontinuity in the velocity field. If we insist on idealized nonviscous flow, the tangential velocity could differ on the top and bottom sides of the trailing edge. In the theory of nonviscous fluids, such a discontinuity can be eliminated either by permitting the flow to round the sharp corner with an infinite velocity gradient or by introducing an exact amount of circulation so that the trailing edge becomes a stagnation point. The latter condition was proposed by the German mathematician Kutta (1902) and the Russian mathematician Joukowski (1907) and is known as *Kutta-Joukowski hypothesis*, which is the basis for our modern theory of flight. Thus, we see that the fluid viscosity, no matter how small, has a profound influence on flow.

But how can we believe the no-slip condition? On what basis is this condition established? The molecular theory of gases does not provide a firm answer. From the molecular hypotheses, Navier deduced (1823) the boundary condition $\beta u = \mu\, \partial u/\partial n$ for flow over a solid wall, where u is the velocity, $\partial u/\partial n$ is the derivative along the normal away from the wall, β is a constant, and μ is the coefficient of viscosity. The ratio μ/β is a length that is zero if there is no slip. Maxwell (1879) calculated that μ/β is a moderate multiple of the mean free path L of the gas molecule—probably about $2L$. This result is in agreement with modern experimental evidence. Since the mean free path of the molecules of the air on the surface of the earth at room temperature is about 5×10^{-8} m, we can say that the no-slip condition may be questioned for micromachines with dimensions on the order of 10^{-6} m; and certainly will not apply to nanomachines, whose dimension is in the nanometer range.

Experiments on the flow of liquids and gases at atmospheric pressure over cm-sized bodies support the no-slip condition conclusively. Coulomb (1800) found that the resistance of an oscillating metallic disk in water was scarcely altered when the disk was smeared with grease or when the surface was covered with powdered sandstone, so the nature of the surface had little influence on the resistance. Poiseuille (1841) and Hagen (1839) obtained precise data on water flow in capillary tubes with diameter on the order of 10–20 μm. Stokes showed that the theoretical result based on the no-slip condition agreed with Poiseuille's experimental results. Other experimenters, such as Whetham (1890) and Couette (1890), came to the same conclusion. Fage and Townsend (1932) used an ultramicroscope to examine the flow of water containing small particles and confirmed the no-slip condition. In addition, there is agreement between theory and experiment on Stokes's and Oseen's theories of motion at small Reynolds numbers, as well as on Taylor's calculations and observations on the stability of flow between rotating cylinders. All these experiences, taken together, support the conclusion that for a liquid, the slip, if it takes place on a solid boundary, is too small to be observed or to make any sensible difference in the results of theoretical deductions.

11.3 SURFACE TENSION AND THE BOUNDARY CONDITIONS AT AN INTERFACE BETWEEN TWO FLUIDS

An interface between two fluids may be regarded as a membrane which has a specific chemical composition and mechanical properties. For example, the surface of a soap bubble in air has a layer of surfactants. The surfaces of pulmonary alveoli have a layer of fluid with surfactants that reduce the surface tension between the lung tissue and the lung gas. A cholesterol vesicle may have a single layer of lipid molecules on its surface or a lipid bilayer. Cell membranes are lipid bilayers. Even at the free surface of water in air, the water molecules at the interface are not in the same state as those in the bulk, and the interface can be regarded as a layer of different material. Hence, if one studies the flow of two fluids separated by an interface, the boundary conditions of the fluids at the interface must take the properties of the interface into consideration.

A membrane is a very thin plate. The stresses in a plate have been discussed in Example 4 of Sec. 1.11, see Fig. 1.6. If the membrane is very thin, we are interested more on the resultant force per unit length in the membrane than in the distribution of stress in its thickness. In thin membrane, the product of the average stress in the membrane and the thickness is called a *stress resultant*, or a *surface tension*, which has the units of [force/length].

Consider a soap bubble in the air, as shown in Fig. 11.2. It is a layer of liquid bounded by two air-liquid interfaces which have surface tension. Assume that the

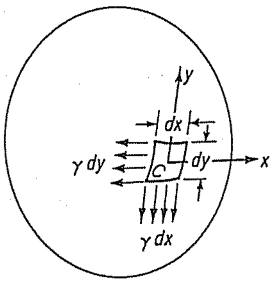

Figure 11.2 A soap bubble.

surface tension is isotropic. Denote the resultant of the surface tensions of the two interfaces by γ. To create the bubble one must blow and create an internal pressure p_i greater than the external pressure p_o, and the force due to the pressure difference must be balanced by the tension in the soap film. Let C be a small, closed rectangular curve of sides dx and dy drawn on the surface of the bubble (Fig. 11.2). The tensions acting on the sides of C are shown in the figure. To compute the pressure required to balance the tensions, let us consider two cross-sectional views: one in the xz-plane (z being normal to the soap film), and another in the yz-plane. The former is shown in Fig. 11.3, where the tensile forces $\gamma\, dy$ act at each end. Since these forces are tangent to the surface, they have a resultant $\gamma\, dy\, d\theta$ normal to

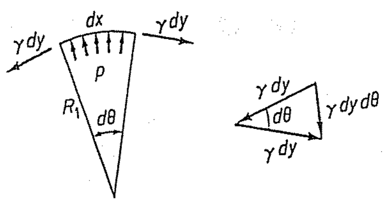

Figure 11.3 Equilibrium of membrane forces acting on an element of the soap bubble.

the surface. But $d\theta = dx/R_1$, where R_1 is the radius of curvature for the soap film. Hence, the normal force is $\gamma\, dx\, dy/R_1$. Similarly, the tensions acting on the other sides of the rectangle contribute a resultant $\gamma\, dx\, dy/R_2$. Since the soap film has two air-liquid interfaces (inside and outside), the total resultant force due to surface tension acting on the curve C is normal to the soap film and is equal to $2\gamma\, dx\, dy/R_1 + 2\gamma\, dx\, dy/R_2$. This force is balanced by the pressure difference multiplied by the area $dx\, dy$. On equating these forces, we obtain, for the soap film, the celebrated equation named after Laplace (1805), although it was actually obtained a year earlier by Thomas Young (1804):

$$2\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = p_i - p_o \tag{11.3-1}$$

If the soap bubble is spherical, then $R_1 = R_2$. If the bubble is not spherical, we note that the sum

$$\frac{1}{R_1} + \frac{1}{R_2} = \text{mean curvature} \tag{11.3-2}$$

is invariant with respect to the rotation of coordinates on any surface. Hence, the directions chosen for the x- and y-axes are immaterial.

As a particular case, let us consider soap films formed by boundary curves under zero pressure difference. Then the surface is the so-called *minimal surface*, governed by the equation

$$\frac{1}{R_1} + \frac{1}{R_2} = 0. \tag{11.3-3}$$

Equation (11.3–1) indicates that the pressure difference required to balance the surface tension becomes very large if the radii R_1 and R_2 become very small. For a constant γ, if $R_1, R_2 \to 0$, the pressure difference tends toward infinity.

If the fluids are moving and the interface is nonstationary, then the no-slip condition must apply in each fluid relative to the interface if the fluid is real (viscous). If one of the fluids is ideal (nonviscous), then there is no no-slip condition for that fluid. If both fluids are ideal, then there is no restriction on slip.

In the most general case for an interface with a specific surface viscosity, surface compressibility, elasticity, and bending rigidity, the equations of motion (or equilibrium) and continuity of the interface are those of thin membranes or thin shells in solid mechanics. The boundary conditions of the fluids in contact with the interface are the nonpenetration and no-slip conditions.

Surface tension is very important in such chemical engineering problems as foaming, in such mechanical engineering problems as the fracture of metals and rocks, and in such biological problems as the opening and collapse of the lung. Surface tension is variable in general. For example, the alveolar surface in our lungs is moist, and the surface tension is modulated by the presence of "surfactants," lipids such as lecithin. The arrangement of these polar molecules on the interface depends on the concentration of the molecules, the rate at which the surface is strained, and the history of strain, so that the surface tension–area relationship has a huge hysteresis loop when the surface is subjected to a periodic strain. Figure 11.4 gives the experimental results obtained by J. A. Clements by means of a surface balance of the Wilhelmy type. Shown are the surface tension–area relationships between air on the one hand and pure water, blood plasma, 1% Tween 20 detergent, and a saline extract of a normal lung on the other. The loops of water and detergent are exaggerated schematically to show the cylic nature of the strain history.

When there is an interface, there is a question of permeability of the fluid moving through it. The permeability will govern the boundary condition with respect to the normal component of velocity. A certain amount of mass transfer, laminar or turbulent mixing, etc., may occur at the interface.

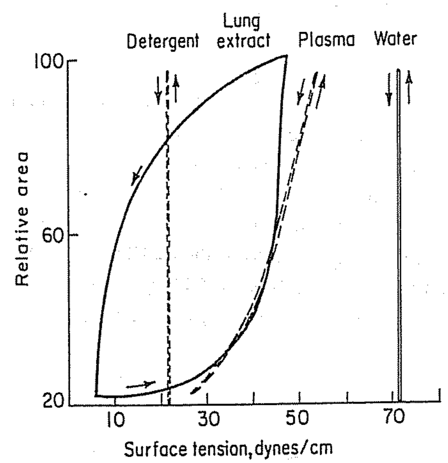

Figure 11.4 The variation in surface tension with strain for several fluids. From J. A. Clements, "Surface Phenomena in Relation to Pulmonary Function," *The Physiologist*, 5(1) (1962), 11–28.

11.4 DYNAMIC SIMILARITY AND REYNOLDS NUMBER

Let us put the Navier-Stokes equation in dimensionless form. For simplicity, we shall consider a homogeneous incompressible fluid. Choose a characteristic velocity V and a characteristic length L. For example, if we investigate the flow of air around an airplane wing, we may take V to be the airplane speed and L to be the wing chord length. If we investigate the flow in a tube, V may be taken as the mean flow speed and L the tube diameter. For a falling sphere, we may take the speed of falling to be V, the diameter of the sphere to be L, and so on. Having chosen these characteristic quantities, we introduce the dimensionless variables

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L}, \quad u' = \frac{u}{V}, \quad v' = \frac{v}{V}, \quad w' = \frac{w}{V}, \quad p' = \frac{p}{\rho V^2}, \quad t' = \frac{Vt}{L} \tag{11.4-1}$$

and the parameter

$$\text{Reynolds number} = R_N = \frac{VL\rho}{\mu} = \frac{VL}{\nu}. \tag{11.4-2}$$

Equation (11.1–7) for an incompressible fluid can then be put into the form

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z} = -\frac{\partial p'}{\partial x'} + \frac{1}{R_N}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right) \tag{11.4-3}$$

and two additional equations obtainable from Eq. (11.4–3) by changing u' into v', v' into w', w' into u' and x' into y', y' into z', z' into x'. The equation of continuity, Eq. (11.1–5), can also be put in dimensionless form:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0. \tag{11.4-4}$$

Since Eqs. (11.4–3) and (11.4–4) constitute the complete set of field equations for an incompressible fluid, it is clear that only one physical parameter, the Reynolds number R_N, enters into the field equations of the flow.

Consider two geometrically similar bodies immersed in a moving fluid under identical initial and boundary conditions. One body may be considered a prototype and the other, a model. The bodies are similar (same shape but different size), and the boundary conditions are identical (in the dimensionless variables). The two flows will be identical if the Reynolds numbers for the two bodies are the same, because two geometrically similar bodies having the same Reynolds number will be governed by identical differential equations and boundary conditions (in dimensionless form). Therefore, *flows about geometrically similar bodies at the same Reynolds numbers are completely similar in the sense that the functions* $u'(x', y', z', t')$, $v'(x', y', z', t')$, $w'(x', y', z', t')$, $p'(x', y', z', t')$ *are the same for the various flows*. This kind of similarity of flows is called *dynamic similarity*. Reynolds number governs dynamic similarity of steady flows. For unsteady flows the requirement for the simulation of the differential equation and the initial and boundary conditions may require the simulation of other dimensionless parameters.

The Reynolds number expresses the ratio of the inertial force to the shear stress. In a flow, the inertial force due to convective acceleration arises from terms such as ρu^2, whereas the shear stress arises from terms such as $\mu\, \partial u/\partial y$. The orders of magnitude of these terms are, respectively,

$$\text{inertial force:} \quad \rho V^2,$$

$$\text{shear stress:} \quad \frac{\mu V}{L}.$$

The ratio is

$$\frac{\text{inertial force}}{\text{shear stress}} = \frac{\rho V^2}{\mu V/L} = \frac{\rho VL}{\mu} = \text{Reynolds number}. \tag{11.4-5}$$

A large Reynolds number signals a preponderant inertial effect. A small Reynolds number signals a predominant shear effect.

The wide range of Reynolds numbers that occurs in practical problems is illustrated in the following examples.

PROBLEMS

11.1 Smokestacks are known to sway in the wind if they are not rigid enough. The wind force depends on the Reynolds number of the flow. Let the wind speed be 30 mi/hr (each mi/hr = 0.44704 m/sec) and the smokestack diameter be 20 ft (each ft = 0.3048 m). Compute the Reynolds number of the flow.

Answer: 5.46×10^6.

The coefficient of viscosity of air at 20°C is $\mu = 1.808 \times 10^{-4}$ poise (g/cm sec), and the kinematic viscosity ν is 0.150 Stoke (cm²/sec).

11.2 Compute the Reynolds number for a submarine periscope of diameter 16 in at 15 knots.

Answer: 2.4×10^6.

For water at 10°C, $\mu = 1.308 \times 10^{-2}$ g/cm sec and $\nu = 1.308 \times 10^{-2}$ cm²/sec. 1 knot = 1 nautical mile per hour, or 1.852 km/hr.

11.3 Suppose that in a cloud chamber experiment designed to determine the charge of an electron (Robert Millikan's experiment), the water droplet diameter is 5 micra (i.e., 5×10^{-4} cm). The droplet moves in air at 0°C at a speed of 2 mm/sec. What is the Reynolds number?

Answer: 7.6×10^{-4}

For air at 0°C, $\nu = 0.132$ cm²/sec.

11.4 For blood plasma to flow in a capillary blood vessel of diameter 10 micra (i.e., 10^{-3} cm) at a speed of 2 mm/sec, what is the Reynolds number?

Answer: 1.4×10^{-2}.

For blood plasma at body temperature, μ is about 1.4 centipoises (1.4×10^{-2} g/cm sec).

11.5 Compute the Reynolds number for a large airplane wing with a chord length of 10 ft (3.048 m), flying at 600 mi/hr (268.224 m/s) at an altitude of 7,500 ft (2,286 m), (0°C).

Answer: 6.2×10^7.

11.5 LAMINAR FLOW IN A HORIZONTAL CHANNEL OR TUBE

Navier-Stokes equations are not easy to solve. If, however, one can find a special problem in which the nonlinear terms disappear, then the solution can be obtained easily sometimes. A particularly simple problem of this nature is the steady flow of an incompressible fluid in a horizontal channel of width $2h$ between two parallel planes, as shown in Fig. 11.5.

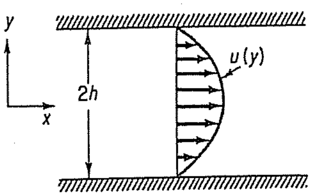

Figure 11.5 Laminar flow in a parallel channel.

We search for a flow

$$u = u(y), \qquad v = 0, \qquad w = 0 \tag{11.5-1}$$

that satisfies the Navier-Stokes equations, the equation of continuity, and the no-slip conditions on the boundaries $y = \pm h$:

$$u(h) = 0, \qquad u(-h) = 0. \tag{11.5-2}$$

Obviously, Eq. (11.5–1) satisfies the equation of continuity, Eq. (11.1–3), exactly, whereas Eq. (11.1–7) becomes

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{d^2u}{dy^2}, \tag{11.5-3}$$

$$0 = \frac{\partial p}{\partial y}, \tag{11.5-4}$$

$$0 = \frac{\partial p}{\partial z}. \tag{11.5-5}$$

Equations (11.5–4) and (11.5–5) show that p is a function of x only. If we differentiate Eq. (11.5–3) with respect to x and use Eq. (11.5–1), we obtain $\partial^2 p/\partial x^2 = 0$. Hence, $\partial p/\partial x$ must be a constant, say, $-\alpha$. Equation (11.5–3) then becomes

$$\frac{d^2u}{dy^2} = -\frac{\alpha}{\mu}, \tag{11.5-6}$$

which has a solution

$$u = A + By - \frac{\alpha}{\mu}\frac{y^2}{2}. \tag{11.5-7}$$

The two constants A and B can be determined by the boundary conditions (11.5–2) to yield the final solution,

$$u = \frac{\alpha}{2\mu}(h^2 - y^2). \tag{11.5-8}$$

Thus, the velocity profile is a parabola.

A corresponding problem is the flow through a horizontal circular cylindrical tube of radius a. (See Fig. 11.6.) We search for a solution

$$u = u(y, z), \qquad v = 0, \qquad w = 0.$$

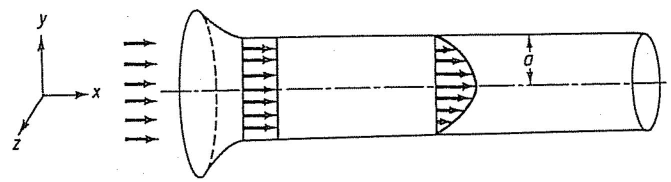

Figure 11.6 Laminar flow in a circular cylindrical tube.

In analogy with Eq. (11.5–6), the Navier-Stokes equation becomes

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\alpha}{\mu}. \tag{11.5–9}$$

It is convenient to transform from the Cartesian coordinates x, y, z to the cylindrical polar coordinates x, r, θ, with $r^2 = y^2 + z^2$. (See Sec. 5.8.) Then Eq. (11.5–9) becomes

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\alpha}{\mu}. \tag{11.5–10}$$

Let us assume that the flow is symmetric, so that u is a function of r only; then $\partial^2 u/\partial\theta^2 = 0$, and the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\frac{\alpha}{\mu} \tag{11.5–11}$$

can be integrated immediately to yield

$$u = -\frac{\alpha}{\mu}\frac{r^2}{4} + A\log r + B. \tag{11.5–12}$$

The constants A and B are determined by the conditions of no slip at $r = a$ and symmetry on the centerline, $r = 0$:

$$u = 0 \quad \text{at} \quad r = a. \tag{11.5–13}$$

$$\frac{du}{dr} = 0 \quad \text{at} \quad r = 0. \tag{11.5–14}$$

The final solution is

$$u = \frac{\alpha}{4\mu}(a^2 - r^2). \tag{11.5–15}$$

This is the famous parabolic velocity profile of the Hagen-Poiseuille flow; the theoretical solution was worked out by Stokes.

The classical solution of the Hagen-Poiseuille flow has been subjected to innumerable experimental observations. It is not valid near the entrance to a tube. It is satisfactory at a sufficiently large distance from the entrance, but is again invalid if the tube is too large or if the velocity is too high. The difficulty at the entry region is due to the transitional nature of the flow in that region, so that our assumption that $v = 0$ and $w = 0$ is not valid. The difficulty with too large a Reynolds number, however, is of a different kind: The flow becomes turbulent!

Osborne Reynolds demonstrated the transition from laminar to turbulent flow in a classical experiment in which he examined an outflow through a small tube from a large water tank. At the end of the tube, he used a stopcock to vary the speed of water through the tube. The junction of the tube with the tank was nicely rounded, and a filament of colored ink was introduced at the mouth. When the

speed of water was slow, the filament remained distinct through the entire length of the tube. When the speed was increased, the filament broke up at a given point and diffused throughout the cross section (see Fig. 11.7). Reynolds identified the governing parameter $u_m d/\nu$—the Reynolds number—where u_m is the mean velocity, d is the diameter, and ν is the kinematic viscosity. Reynolds found that the transition from laminar to turbulent flow occurred at Reynolds numbers btween 2,000 and 13,000, depending on the smoothness of the entry conditions. When extreme care is taken, the transition can be delayed to Reynolds numbers as high as 40,000. On the other hand, a value of 2,000 appears to be about the lowest value obtainable on a rough entrance.

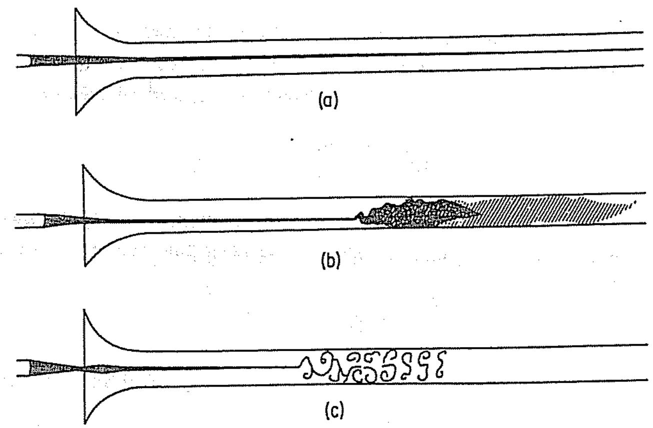

Figure 11.7 Reynolds's turbulence experiment: (a) laminar flow; (b) and (c), transition from laminar to turbulent flow. After Osborne Reynolds, "An Experimental Investigation of the Circumstances which Determine whether the Motion of Water Shall Be Direct or Sinuous, and of the Law of Resistance in Parallel Channels, *Phil. Trans., Roy. Soc.*, 174 (1883), 935–982.

Turbulence is one of the most important and most difficult problems in fluid mechanics. It is technically important not only because turbulence affects skin friction, resistance to flow, heat generation and transfer, diffusion, etc., but also because it is widespread. One might say that the normal mode of fluid flow is turbulent. The water in the ocean, the air above the earth, and the state of motion in the sun are turbulent. The theory of turbulence will greet you wherever you turn when you study fluid mechanics in greater depth.

PROBLEM

11.6 From the basic solution given by Eq. (11.5–15), show that the rate of mass flow through the tube is

$$Q = \frac{\pi a^4 \rho}{8\mu}\alpha, \tag{11.5–16}$$

that the mean velocity is

$$u_m = \frac{a^2}{8\mu}\alpha, \tag{11.5–17}$$

and that the skin friction coefficient is

$$c_f = \frac{\text{shear stress}}{\text{mean dynamic pressure}} = \frac{-\mu(\partial u/\partial r)_{r=a}}{\frac{1}{2}\rho u_m^2} = \frac{16}{R_N}, \tag{11.5–18}$$

where $R_N = 2au_m/\nu$.

11.6 BOUNDARY LAYER

If we let $R_N \to \infty$ in the dimensionless Navier-Stokes equation (11.4–3) for a homogeneous incompressible fluid, namely,

$$\frac{Du_i'}{Dt'} = -\frac{\partial p'}{\partial x_i'} + \frac{1}{R_N}\nabla^2 u_i', \qquad (i = 1, 2, 3), \tag{11.6–1}$$

the last term would drop out unless the second derivatives become very large. In a general flow field in which the velocity and its derivatives are finite, the effect of viscosity would disappear when the Reynolds number tends toward infinity. Near a solid wall, however, a rapid transition takes place for the velocity to vary from that of the free stream to that of the solid, because of the no-slip condition. If this transition layer is very thin, the last term cannot be dropped, even though the Reynolds number is very large.

We shall define the boundary layer as the region of a fluid in which the effect of viscosity is felt, even though the Reynolds number is very large. In the boundary layer, the flow is such that the shear-stress term—the last term in Eq. (11.6–1)—is of the same order of magnitude as the convective force term. Based on the observation that in a high-speed flow the boundary layer is very thin, Prandtl (1904) simplified the Navier-Stokes equation into a much more tractable boundary-layer equation.

To see the nature of the boundary-layer equation, let us consider a two-dimensional flow over a fixed flat plate. (See Fig. 11.8.) We take the x'-axis in the direction of flow along the surface and the y'-axis normal to it. The velocity component w along the z'-axis is assumed to vanish. Then Eq. (11.6–1) becomes

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{R_N}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right), \tag{11.6–2}$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial p'}{\partial y'} + \frac{1}{R_N}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right). \tag{11.6–3}$$

If we take the free-stream velocity as the characteristic velocity, then the dimensionless velocity u' is equal to 1 in the free stream (outside the boundary layer).

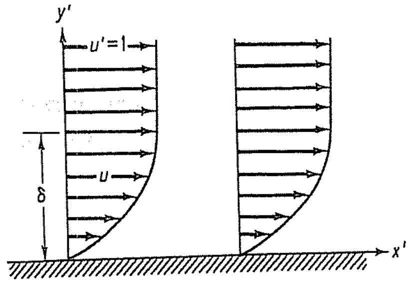

Figure 11.8 A boundary layer of flow.

The velocity u' varies from 0 on the solid surface $y' = 0$ to 1 at $y' = \delta$, where δ denotes the boundary-layer thickness (which is dimensionless and numerically small). We can now estimate the order of magnitude of the terms occurring in Eq. (11.6–2) as follows. We write $u' = O(1)$ to mean that u' is *at most* on the order of unity. We notice that $O(1) + O(1) = O(1)$, $O(1)\cdot O(1) = O(1)$, $O(1) + O(\delta) = O(1)$, and $O(1)\cdot O(\delta) = O(\delta)$. Then, since the variation of u' with respect to t' and x' is finite, we have

$$u' = O(1), \quad \frac{\partial u'}{\partial x'} = O(1), \qquad \frac{\partial^2 u'}{\partial x'^2} = O(1), \quad \frac{\partial u'}{\partial t'}\, O(1). \tag{11.6–4}$$

By the equation of continuity, Eq. (11.4–4), we have

$$\frac{\partial u'}{\partial x'} = -\frac{\partial v'}{\partial y'} = O(1). \tag{11.6–5}$$

Hence,

$$v' = \int_0^\delta \frac{\partial v'}{\partial y'}dy' \sim \int_0^\delta O(1)dy = O(\delta). \tag{11.6–6}$$

Thus, the vertical velocity is at most on the order of δ, which is numerically small:

$$\delta \ll 1. \tag{11.6–7}$$

Since $v' = O(\delta)$ while $\partial v'/\partial y' = O(1)$ according to Eq. (11.6–5), we see that a differentiation of a quantity with respect to y' in the boundary layer increases the order of magnitude of that quantity by $1/\delta$. Then

$$\frac{\partial u'}{\partial y'} = O\left(\frac{1}{\delta}\right), \quad \frac{\partial^2 u'}{\partial y'^2} = O\left(\frac{1}{\delta^2}\right), \qquad \frac{\partial v'}{\partial x'} = O(\delta), \quad \frac{\partial^2 v'}{\partial x'^2} = O(\delta), \qquad \frac{\partial v'}{\partial t'} = O(\delta), \quad \frac{\partial^2 v'}{\partial y'^2} = O\left(\frac{1}{\delta}\right). \tag{11.6–8}$$

Now, by definition, the shear stress term is of the same order of magnitude as the inertial force term in the boundary layer. But the terms on the left-hand side of Eq. (11.6–2) are all $O(1)$; hence, those on the right-hand side must be also $O(1)$; in particular,

$$O(1) = \frac{\partial p'}{\partial x'},$$
$$O(1) = \frac{1}{R_N}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) = \frac{1}{R_N}\left[O(1) + O\left(\frac{1}{\delta^2}\right)\right]. \qquad (11.6\text{–}9)$$

Since the first term in the bracket is much smaller than the second term, we have

$$O(1) = \frac{1}{R_N} O\left(\frac{1}{\delta^2}\right).$$

Hence,

$$R_N = O\left(\frac{1}{\delta^2}\right). \qquad (11.6\text{–}10)$$

Thus, we obtain an estimate of the boundary-layer thickness:

$$\delta = O\left(\frac{1}{\sqrt{R_N}}\right). \qquad (11.6\text{–}11)$$

Substituting Eqs. (11.6–4), (11.6–8), and (11.6–10) into Eq. (11.6–3), we see that all terms involving v' are $O(\delta)$; hence, the remaining term $\partial p'/\partial y'$ must also be $O(\delta)$. Thus,

$$\frac{\partial p'}{\partial y'} = O(\delta) \sim 0. \qquad (11.6\text{–}12)$$

In other words, *the pressure is approximately constant through the boundary layer.* By retaining only terms of order 1, the Navier-Stokes equations are reduced to

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{R_N}\frac{\partial^2 u'}{\partial y'^2}, \qquad (11.6\text{–}13)$$

and Eq. (11.6–12). Equation (11.6–13) is Prandtl's boundary-layer equation; it is subjected to the boundary conditions

$$u' = v' = 0 \quad \text{for } y' = 0, \qquad (11.6\text{–}14)$$
$$u' = 1 \quad \text{for } y' = \delta.$$

PROBLEM

11.7 Estimate the boundary-layer thickness of air flowing over a plate 10 ft (3.048 m) long at 100 ft/sec (30.48 m/s).

Answer: At 20°C, $\delta = O(4.018 \times 10^{-4})$. With a chord length of 3.048 m, the boundary-layer thickness ≐ 0.12 cm.

11.7 LAMINAR BOUNDARY LAYER OVER A FLAT PLATE

To apply Prandtl's boundary-layer theory, let us consider an incompressible fluid flowing over a flat plate, as in Fig. 11.9, in which the vertical scale is magnified to make the picture clearer. The velocity outside the boundary layer is assumed constant, $\bar{u}$. We shall seek a steady-state solution for which $\partial u/\partial t = 0$. An additional assumption will be made, to be justified *a posteriori*, that the pressure gradient $\partial p/\partial x$ is negligible, compared with the other terms in the boundary-layer equation. Then Eq. (11.6–13) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \qquad (11.7\text{–}1)$$

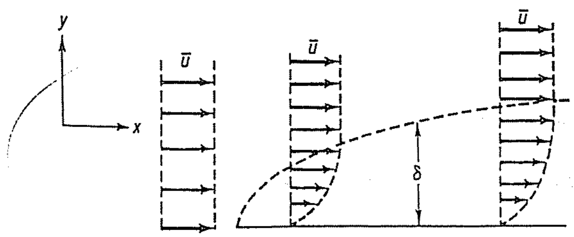

Figure 11.9 Laminar boundary layer over a flat plate, showing the growth in thickness of the boundary layer.

Here we return to the physical quantities and drop the primes. The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (11.7\text{–}2)$$

Equation (11.7–2) is satisfied identically if u, v are derived from a stream function $\psi(x, y)$:

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \qquad (11.7\text{–}3)$$

Then Eq. (11.7–1) becomes

$$\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} = \nu\frac{\partial^3 \psi}{\partial y^3}. \qquad (11.7\text{–}4)$$

The boundary conditions are (a) no slip on the plate and (b) continuity at the free stream outside the boundary layer; i.e.,

$$u = v = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \qquad \text{for } y = 0, \qquad (11.7\text{–}5)$$

$$u = \bar{u} \qquad \text{or} \quad -\frac{\partial \psi}{\partial y} = \bar{u} \qquad \text{for } y = \delta. \tag{11.7–6}$$

Following Blasius,* we seek a "similarity" solution. Consider the transformation

$$\bar{x} = \alpha x, \qquad \bar{y} = \beta y, \qquad \bar{\psi} = \gamma \psi, \tag{11.7–7}$$

in which α, β, and γ are constants. A substitution of Eq. (11.7–7) into Eq. (11.7–4) shows that the equation for the function $\bar{\psi}(\bar{x}, \bar{y})$ has the same form as Eq. (11.7–4) if we choose $\gamma = \alpha/\beta$. A similar substitution into Eq. (11.7–6) shows that $-\partial\bar{\psi}/\partial\bar{y} = \bar{u}$ if we choose $\gamma = \beta$. Hence, $\beta = \alpha/\beta$, or $\beta = \sqrt{\alpha}$. With this choice, we have

$$\frac{\bar{y}}{\sqrt{\bar{x}}} = \frac{y}{\sqrt{x}}, \qquad \frac{\bar{\psi}}{\sqrt{\bar{x}}} = \frac{\psi}{\sqrt{x}}. \tag{11.7–8}$$

These relations suggest that there are solutions of the form

$$\psi = -f(\xi)\sqrt{\nu \bar{u} x}, \qquad \xi = \sqrt{\frac{\bar{u}}{\nu}}\frac{y}{\sqrt{x}}. \tag{11.7–9}$$

Substitution of Eq. (11.7–9) into Eq. (11.7–4) yields the ordinary differential equation

$$2f''' + ff'' = 0, \tag{11.7–10}$$

where the primes indicate differentiation with respect to ξ. This equation has been solved numerically to a high degree of accuracy under the boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 1, \tag{11.7–11}$$

which say that $u = 0$ and $v = 0$ at the plate and $u \to \bar{u}$, the free-stream velocity, outside the boundary layer. From Eq. (11.7–9), it is seen that for fixed x/L, $\xi \to \infty$ means that y/L is large, compared with the boundary-layer thickness $\sqrt{\nu L/\bar{u}}$, or δ. The velocity distribution, yielded by the solution of Eqs. (11.7–10) and (11.7–11), agrees closely with experimental evidence,† as seen in Fig. 11.10, except very near the leading edge of the plate, where the boundary-layer approximation breaks down, and far downstream, where the flow becomes turbulent.

The flow corresponding to the solution given by Eq. (11.7–9), (11.7–10), and (11.7–11) is a laminar flow. At a sufficient distance downstream from the leading edge, the flow becomes turbulent and the Blasius solution fails. The transition occurs when a Reynolds number based on the boundary layer thickness,

$$R = \frac{\bar{u}\delta}{\nu},$$

*H. Blasius, "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Zeitschrift f. Math. u. Phys.*, 56 (1908), 1.

†J. Nikuradse, *Laminare Reibungsschichten an der längsangeströmten platte.* Monograph, Zentrale f. Wiss. Berichtswesen, Berlin, 1942. See H. Schlichting, *Boundary Layer Theory*, translated by J. Kestin, New York: McGraw-Hill Book Company (1960), p. 124.

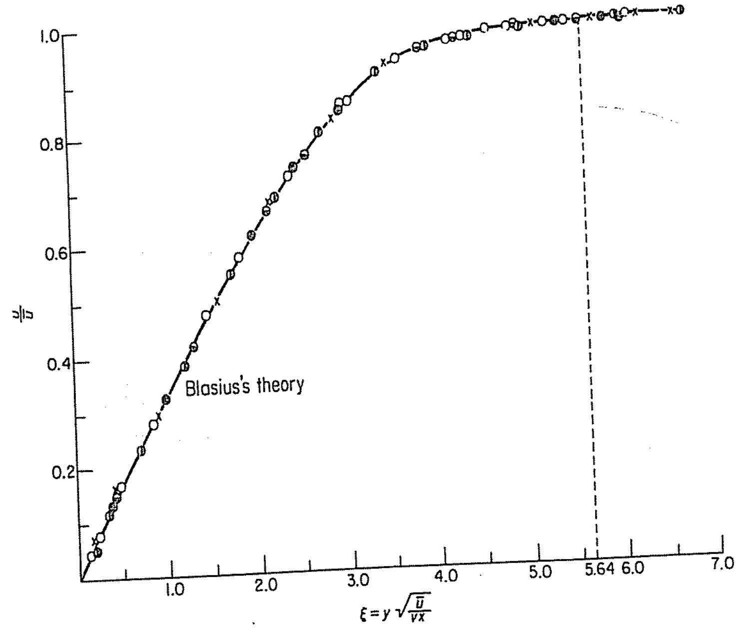

Figure 11.10 Blasius's solution of velocity distribution in a laminar boundary layer on a flat plate at zero incidence and comparison with Nikuradse's measurements.

reaches a critical value. Generally, the value of the critical transitional Reynolds number is on the order of 3,000, but the exact value depends on the surface roughness, curvature, Mach number, etc.

There is a tremendous difference between a laminar boundary layer and a turbulent one with respect to heat transfer, skin friction, heat generation, etc. In our space age, the question of laminar-turbulent transition is of supreme importance for reentry vehicles. As a satellite reenters the atmosphere, the heat generated by skin friction in the boundary layer is tremendous—but a turbulent boundary layer generates much more heat than a laminar one. For most reentry vehicles, survival is possible if the boundary layer over the nose cone is laminar; if the flow became turbulent, the nose cone could be burned out.

11.8 NONVISCOUS FLUID

A great simplification is obtained if the coefficient of viscosity vanishes exactly. Then the stress tensor is isotropic, i.e.,

$$\sigma_{ij} = -p\delta_{ij}, \tag{11.8–1}$$

and the equation of motion can be simplified to

$$\rho\frac{Dv_i}{Dt} = \rho X_i - \frac{\partial p}{\partial x_i}. \tag{11.8–2}$$

Here, ρ is the density of the fluid; p is the pressure; v_1, v_2, v_3 are the velocity components; and X_1, X_2, X_3 are the body force components per unit mass.

If, in addition, the fluid is homogeneous and incompressible, then its density is a constant, and the equation of continuity, Eq. (11.1–3), is reduced to the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{or} \quad \frac{\partial u_i}{\partial x_i} = 0. \tag{11.8–3}$$

A vector field satisfying Eq. (11.8–3) is said to be *solenoidal.* According to the general theory of potentials, a solenoidal field can be derived from another vector field. This can be illustrated in the simple case of a *two-dimensional flow field* for which $w = 0$ and u, v are independent of z and for which the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{11.8–4}$$

Then it is obvious that if we take an arbitrary function $\psi(x, y)$ and derive u, v according to the rules

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{11.8–5}$$

Eq. (11.8–4) will be satisfied identically. Such a function ψ is called a *stream function.*

Substituting Eq. (11.8–5) into the equation of motion, Eq. (11.8–2), we obtain the governing equations (for the two-dimensional flow),

$$\begin{aligned} \frac{\partial^2\psi}{\partial t\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \\ -\frac{\partial^2\psi}{\partial t\partial x} - \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y}. \end{aligned} \tag{11.8–6}$$

If the body force is zero, an elimination of p yields

$$\frac{\partial}{\partial t}\nabla^2\psi + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y = 0, \tag{11.8–7}$$

in which

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

and the subscripts indicate partial differentiation.

PROBLEM

11.8 Show that for a two-dimensional flow of an incompressible viscous fluid, the governing equation for the stream function defined by Eq. (11.8–5) is

$$\frac{\partial}{\partial t}\nabla^2\psi + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y = \nu\nabla^2\nabla^2\psi + \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}. \tag{11.8–8}$$

11.9 VORTICITY AND CIRCULATION

The concepts of circulation and vorticity are of great importance in fluid mechanics. The *circulation* $I(\mathscr{C})$ in any closed circuit $\mathscr{C}$ is defined by the line integral

$$I(\mathscr{C}) = \int_{\mathscr{C}} \mathbf{v}\cdot\mathbf{dl} = \int_{\mathscr{C}} v_i dx_i, \tag{11.9–1}$$

where $\mathscr{C}$ is any closed curve in the fluid and the integrand is the scalar product of the velocity vector $\mathbf{v}$ and the vector $\mathbf{dl}$, which is tangent to the curve $\mathscr{C}$ and of length dl (Fig. 11.11). Clearly, the circulation is a function of both the velocity field and the chosen curve $\mathscr{C}$.

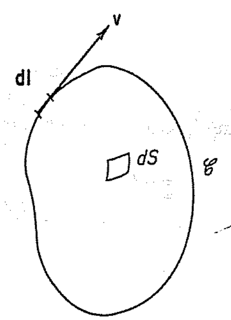

Figure 11.11 Circulation: Notations.

By means of Stokes's theorem, if $\mathscr{C}$ encloses a simply connected region, the line integral can be transformed into a surface integral

$$I(\mathscr{C}) = \int_S (\nabla \times \mathbf{v})_n dS = \int_S (\text{curl } \mathbf{v})_i \nu_i dS, \tag{11.9–2}$$

where S is any surface in the fluid bounded by the curve $\mathscr{C}$, ν_i is the unit normal to the surface, and curl $\mathbf{v} = e_{ijk}v_{j,k}$. The curl $\mathbf{v}$ is called the *vorticity* of the velocity field.

The law of change of circulation with time, when the circuit $\mathscr{C}$ is a *fluid line*, i.e., a curve $\mathscr{C}$ formed by the same set of fluid particles as time changes, is given by the *theorem of Lord Kelvin*: If the fluid is nonviscous and the body force is conservative, then

$$\frac{DI}{Dt} = -\int_{\mathscr{C}} \frac{dp}{\rho}. \tag{11.9–3}$$

If, in addition to the preceding conditions, the density ρ is a unique function of the pressure, then the fluid is called *barotropic*, and the last integral vanishes because the integral would be single valued and $\mathscr{C}$ is a closed curve. We then have the *Helmholtz theorem* that

$$\frac{DI}{Dt} = 0. \tag{11.9–4}$$

To prove the foregoing theorems, we note that since $\mathscr{C}$ is a fluid line composed always of the same particles, the order of differentiation and integration may be interchanged in the following:

$$\frac{D}{Dt}\int_{\mathscr{C}} v_i\,dx_i = \int_{\mathscr{C}} \frac{D}{Dt}(v_i dx_i) = \int_{\mathscr{C}}\left(\frac{Dv_i}{Dt}dx_i + v_i\frac{D\,dx_i}{Dt}\right). \tag{11.9–5}$$

But $D\,dx_i/Dt$ is the rate at which dx_i is increasing as a consequence of the motion of the fluid; hence, it is equal to the difference of the velocities parallel to x_i at the ends of the element, i.e., dv_i. Substituting Dv_i/Dt from the equation of motion, Eq. (11.8–2), and replacing $D\,dx_i/Dt$ by dv_i, we obtain

$$\frac{DI}{Dt} = \int_{\mathscr{C}}\left[\left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + X_i\right)dx_i + v_i\,dv_i\right]$$

$$= -\int_{\mathscr{C}}\frac{dp}{\rho} + \int_{\mathscr{C}} X_i\,dx_i + \int_{\mathscr{C}} dv^2. \tag{11.9–6}$$

Of the terms on the right-hand side, the last vanishes because v^2 is single valued in the flow field; the second vanishes if the body force X_i is conservative. Hence, Kelvin's theorem is proved. Helmholtz's theorem follows immediately as a special case because the integral on the right-hand side vanishes if the fluid is barotropic.

In the clear-cut conclusion of Helmholtz's theorem lies its importance. For if we limit our attention to a barotropic fluid, then we have I = const. Hence, if the circulation vanishes at one instant of time, it must vanish for all times. If this is so for any arbitrary fluid lines in a field, then, according to Eq. (11.9–2), the vorticity vanishes in the whole field. This leads to a great simplification, which will be discussed in Sec. 11.10, namely, the irrotational flow. To appreciate the importance of this simplification, one need observe only that a vast majority of the classical literature on fluid mechanics deals with irrotational flows.

Note that the circulation around a fluid line *does not* have to remain constant if the density ρ depends on other variables in addition to pressure. Into this category fall most geophysical problems in which the temperature enters as a parameter affecting both ρ and p. Also, in stratified flows, ρ is a function of location, not necessarily a function of p alone.

The significance of the term *fluid line* in the theorems of Kelvin and Helmholtz may be seen by considering the problem of a thin airfoil moving in the air. The conditions of the Helmholtz theorem are satisfied. Hence, the circulation I about any fluid line never changes with time. Since the motion of the fluid is caused by the motion of the airfoil, and since at the beginning the fluid is at rest and $I = 0$, it follows that I vanishes at all times. Note, however, that the volume occupied by the airfoil is exclusive of the fluid. A fluid line $\mathscr{C}$ enclosing the boundary of the airfoil becomes elongated when the airfoil moves forward, as shown in Fig. 11.12. According to the Helmholtz theorem, the circulation about $\mathscr{C}$ is zero, so that the total vorticity inside $\mathscr{C}$ vanishes, but one cannot conclude that the vorticity actually vanishes everywhere inside $\mathscr{C}$. In the region occupied by the airfoil *and* in the wake behind the airfoil, vorticity does exist. However, the Helmholtz theorem applies to the region outside the airfoil and its wake, and the vanishing of circulation about every possible fluid line shows clearly that the flow is irrotational outside the airfoil *and* its wake.

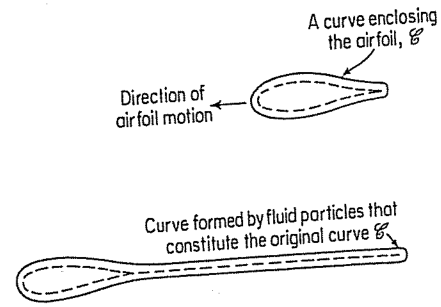

Figure 11.12 Fluid line enclosing an airfoil and its wake.

11.10 IRROTATIONAL FLOW

A flow is said to be *irrotational* if the vorticity vanishes everywhere, i.e., if

$$\nabla \times \mathbf{v} = \text{curl } \mathbf{v} = 0, \tag{11.10–1}$$

or

$$e_{ijk}v_{j,k} = 0.$$

For a two-dimensional irrotational flow, we must have

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \tag{11.10–2}$$

If the fluid is incompressible and a stream function defined by Eq. (11.8–5) is introduced, then a substitution of Eq. (11.8–5) into Eq. (11.10–2) yields the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{11.10–3}$$

This is the famous *Laplace equation*, whose solution is the concern of many books on applied mathematics.

We can show that an irrotational flow of an incompressible fluid is governed by a Laplace equation even in the three-dimensional case, because by the definition of irrotationality, the following three equations must hold:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0, \qquad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0. \tag{11.10–4}$$

These equations can be satisfied identically if the velocities u, v, w are derived from a *potential function* $\Phi(x, y, z)$ according to the rule

$$u = \frac{\partial \Phi}{\partial x}, \qquad v = \frac{\partial \Phi}{\partial y}, \qquad w = \frac{\partial \Phi}{\partial z}. \tag{11.10–5}$$

If, in addition, the fluid is *incompressible*, then a substitution of Eq. (11.10–5) into Eq. (11.1–5) yields the *Laplace equation*

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \tag{11.10–6}$$

Since Φ is a potential function, this equation is also called a *potential equation*.

The incompressible potential flow is governed by the Laplace equation. If a solution can be found that satisfies all the boundary conditions, then the Eulerian equation of motion yields the pressure gradient, and the problem is solved. The nonlinear convective acceleration, which causes the central difficulty of fluid mechanics, does not hinder the solution of potential flows of an incompressible fluid. This is why the potential theory is so simple and so important.

To realize the usefulness of the potential theory, we quote the Helmholtz theorem (see Sec. 11.9): If the motion of any portion of a fluid mass is irrotational at any one instant of time, it will continue to be irrotational at all times, provided that the body forces are conservative and that the fluid is *barotropic* (i.e., its density is a function of pressure alone). These conditions are met in many problems. If a solid body is immersed in a fluid and suddenly set in motion, the motion generated in a nonviscous fluid is irrotational.* Hence, a whole class of technologically important problems is irrotational.

11.11 COMPRESSIBLE NONVISCOUS FLUIDS

Basic Equations

If a fluid is compressible, the equation of continuity, Eq. (10.5–3), is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0. \tag{11.11–1}$$

*See H. Lamb, *Hydrodynamics*, New York: Dover Publications, 6th ed. (1945), pp. 10, 11.

If the fluid is nonviscous, the Eulerian equation of motion is

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + X_i. \tag{11.11–2}$$

The density is uniquely related to the pressure only if the temperature T is explicitly accounted for. Thus, if the temperature is known to be constant (isothermal), we have, for an ideal gas,

$$\frac{p}{\rho} = \text{const.}, \qquad T = \text{const.}, \tag{11.11–3}$$

whereas if the flow is isentropic (adiabatic and reversible), we have

$$\frac{p}{\rho^{\gamma}} = \text{const.}, \qquad \frac{T}{\rho^{\gamma-1}} = \text{const.}, \tag{11.11–4}$$

where γ is the ratio of the specific heats of the gas at constant pressure, C_p, and constant volume, C_v; i.e., $\gamma = C_p/C_v$. Both cases are *barotropic*.

In other cases, it is necessary to introduce the temperature explicitly as a variable. Then we must introduce also the equation of state relating p, ρ, and T and the *caloric* equation of state relating C_p, C_v, and T.

Small Disturbances

Let us consider, as an example, the propagation of small disturbances in a barotropic fluid in the absence of body force. Let us write

$$c^2 = \frac{dp}{d\rho}. \tag{11.11–5}$$

The velocity of flow will be assumed to be so small that the second-order terms may be neglected in comparison with the first-order term. Correspondingly, the disturbances in the density ρ and the pressure p and the derivatives of ρ and p are also first-order infinitesimal quantities. Then, on neglecting the body force X_i and all small quantities of the second or higher order, Eqs. (11.11–1) and (11.11–2) are linearized to

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_j}{\partial x_j} = 0, \tag{11.11–6}$$

$$\frac{\partial v_i}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} = -\frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial x_i} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x_i}. \tag{11.11–7}$$

Differentiating Eq. (11.11–6) with respect to t and Eq. (11.11–7) with respect to x_i, again neglecting the second-order terms, and eliminating the sum $\rho\, \partial^2 v_j/\partial t\, \partial x_j$, we obtain

$$\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho}{\partial x_i \partial x_i}; \tag{11.11–8}$$

i.e.,

$$\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}.$$

This is the *wave equation for the propagation of small disturbances*. It is the basic equation of acoustics.

By the same linearization procedure, and because the change in pressure is proportional to the change in density, $dp = c^2\, d\rho$, we see that the same wave equation governs the pressure disturbance:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x_k \partial x_k}. \tag{11.11–9}$$

Further, from Eqs. (11.11–7) and (11.11–8) or (11.11–9), we deduce that

$$\frac{1}{c^2}\frac{\partial^2 v_i}{\partial t^2} = \frac{\partial^2 v_i}{\partial x_k \partial x_k}. \tag{11.11–10}$$

Hence, in the linearized theory, ρ, p, v_1, v_2, and v_3 are governed by the same wave equation.

Propagation of Sound

Let us apply these equations to the problem of a source of disturbance (sound) located at the origin and radiating symmetrically in all directions. We may visualize a spherical siren. Because of the radial symmetry, we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}. \tag{11.11–11}$$

Hence, Eq. (11.11–8) becomes

$$\frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho}{\partial r^2} + \frac{2}{r}\frac{\partial \rho}{\partial r}. \tag{11.11–12}$$

It can be verified by direct substitution that a general solution of this equation is the sum of two arbitrary functions f and g:

$$\rho = \rho_0 + \frac{1}{r} f(r - ct) + \frac{1}{r} g(r + ct), \tag{11.11–13}$$

Here, ρ_0 is a constant (the undisturbed density of the field), the f term represents a wave radiating out from the origin, and the g term represents a wave converging toward the origin. Perhaps the clearest way to see this is to consider a special case in which the function $f(r - ct)$ is a step function: $f(r - ct) = \epsilon \mathbb{1}(r - ct)$, where ϵ

is small and $\mathbb{1}(r - ct)$ is the unit-step function, which is zero when $r - ct < 0$ and is 1 when $r - ct > 0$. The disturbance is, therefore, a small jump across a line of discontinuity described by the equation $r - ct = 0$. At time $t = 0$, the disturbance is located at the origin. At time t, the line of discontinuity is moved to $r = ct$. Thus, c is the speed of propagation of the disturbance. The general case follows by the principle of superposition. In acoustics, c is called the *velocity of sound*.

The velocity of sound $c = (dp/d\rho)^{1/2}$ depends on the relationship between pressure and density. If we are concerned with an ideal gas and the condition is isentropic, we have, from Eq. (11.11–4),

$$c = \sqrt{\frac{\gamma p}{\rho}}. \tag{11.11–14}$$

In the history of mechanics, there was a long story about the propagation of sound in air. The first theoretical investigation of the velocity of sound was made by Newton (1642–1727), who assumed Eq. (11.11–3) and obtained $c = \sqrt{p/\rho}$ in a publication in 1687. It was found that the value calculated from Newton's formula falls short of the experimental value of the speed of sound by a factor of approximately one-sixth. This discrepancy was not explained until Laplace (1749–1827) pointed out that the rate of compression and expansion in a sound wave is so fast that there is no time for any appreciable interchange of heat by conduction; thus, the process must be considered adiabatic. This argument becomes plausible if we think of the step wave discussed in the preceding paragraph. For a step wave, the sudden changes in ρ and p that take place as the wave front sweeps by must be accomplished at the wave front in an infinitesimal region of space and time. Heat transfer in such a small time interval is negligible. Hence, the gas flows isentropically across the discontinuity. As a general sound wave is a superposition of such step waves, the entire flow is isentropic. Therefore, Eq. (11.11–4) applies and Eq. (11.11–14) results. Experiments have verified that Laplace was right.

Generally, then, the wave equations (11.11–8), et seq., are associated with isentropic flows. To apply these equations, conditions that guarantee isentropy, such as the absence of strong shock waves and small thermal diffusivity, must be observed.

11.12 SUBSONIC AND SUPERSONIC FLOW

Basic Equations in Laboratory Frame of Reference

The basic wave equation (11.11–8) is referred to a frame of reference that is at rest relative to the fluid at infinity. The equation imposes no restriction on where and how the disturbances are generated. The sources of disturbances may be moving or changing with time; the same equation holds. The nature of the sources would appear only in the boundary conditions and initial conditions.

A flying aircraft is a source of disturbances in still air. The disturbances come to us as sound waves governed by the wave equation. As we all know, the nature of the disturbances changes drastically as the aircraft's flight speed changes from subsonic to supersonic. In the latter case, we hear the sonic boom.

It is convenient to study the nature of flow about an aircraft in a wind tunnel. We shall therefore write down the wave-propagation equation for disturbances in the air flowing in a tunnel as they appear to us standing on the ground.

Consider a body of fluid coming from, say, the left, with a uniform velocity U at infinity. If the disturbances are indicated by a prime, we assume the velocity components to be

$$u = U + u', \qquad v = v', \qquad w = w', \qquad U = \text{const.} \tag{11.12–1}$$

and the pressure and density

$$p = p_0 + p', \qquad \rho = \rho_0 + \rho'. \tag{11.12–2}$$

The whole investigation would be simplified if we could assume that the disturbances are infinitesimal quantities of the first order; i.e.,

$$u', v', w' \ll U, \qquad p' \ll p_0, \qquad \rho' \ll \rho_0. \tag{11.12–3}$$

Under these assumptions, the basic equations (11.11–1) and (11.11–4) may be linearized as before. In fact, repeating the relevant steps in Sec. 11.11 with our new assumptions, we obtain the equation of continuity

$$\frac{\partial \rho'}{\partial t} + \rho_0\left(\frac{\partial U}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) + (U + u')\frac{\partial \rho'}{\partial x} + v'\frac{\partial \rho'}{\partial y} + w'\frac{\partial \rho'}{\partial z} = 0,$$

which is linearized to

$$\frac{\partial \rho'}{\partial t} + \rho_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) + U\frac{\partial \rho'}{\partial x} = 0. \tag{11.12–4}$$

Similarly, the equations of motion are linearized to

$$\begin{aligned}
\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} = -\frac{c^2}{\rho_0}\frac{\partial \rho'}{\partial x},\\
\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} &= -\frac{c^2}{\rho_0}\frac{\partial \rho'}{\partial y},\\
\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} &= -\frac{c^2}{\rho_0}\frac{\partial \rho'}{\partial z}.
\end{aligned} \tag{11.12–5}$$

Differentiating the three equations (11.12–5) with respect to x, y, and z, respectively, adding, and again neglecting the second-order terms, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) = -\frac{c^2}{\rho_0}\left(\frac{\partial^2 \rho'}{\partial y^2} + \frac{\partial^2 \rho'}{\partial y^2} + \frac{\partial^2 \rho'}{\partial z^2}\right).$$

Hence, on eliminating $\partial u'/\partial x + \partial v'/\partial y + \partial w'/\partial z$ with Eqs. (11.12–4), we have

$$\frac{\partial^2 \rho'}{\partial t^2} + 2U\frac{\partial^2 \rho'}{\partial x \partial t} + U^2\frac{\partial^2 \rho'}{\partial x^2} = c^2\left(\frac{\partial^2 \rho'}{\partial x^2} + \frac{\partial^2 \rho'}{\partial y^2} + \frac{\partial^2 \rho'}{\partial z^2}\right). \tag{11.12–6}$$

This is the basic equation for compressible flow in aerodynamics.

If, to Eq. (11.12–6) we apply the method used in Sec. 11 to derive Eqs. (11.11–9), (11.11–10) from Eq. (11.11–8), we can show that the pressure p' and velocity components v_i' satisfy the same equation. If the flow is irrotational, then the velocity potential Φ, for which $v_i = \Phi,_i$, also satisfies this equation.

Steady Flow

Let us examine the basic equation (11.12–6) in some simpler cases. Consider a steady flow around a model at rest. Then all derivatives with respect to time t vanish, and the velocity potential Φ is governed by the equation

$$\frac{U^2}{c^2}\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}. \tag{11.12–7}$$

This equation now depends on only one dimensionless parameter, U/c, which is called the *Mach number* and is denoted by

$$M = \frac{U}{c}. \tag{11.12–8}$$

The nature of the solution to Eq. (11.12–7) depends on whether M is greater or less than 1. We call a flow *subsonic* if $M < 1$, *supersonic* if $M > 1$. We write, for a subsonic flow,

$$(1 - M^2)\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (M < 1), \tag{11.12–9}$$

whereas, for a supersonic flow, we have

$$(M^2 - 1)\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (M > 1). \tag{11.12–10}$$

Equation (11.12–9) is a partial differential equation of the *elliptic type*. Equation (11.12–10) is one of the *hyperbolic type*. Let us consider an example showing the difference between these equations.

Example: Steady Flow over a Wavy Plate

Let a very thin plate with a small sinusoidal wavy profile be placed in a steady flow, with the mean chord of the plate parallel to the velocity U at infinity. (See Figs. 11.13 and 11.14.) The waves of the plate are described by the equation

$$z = a \sin\frac{\pi x}{L}. \tag{11.12–11}$$

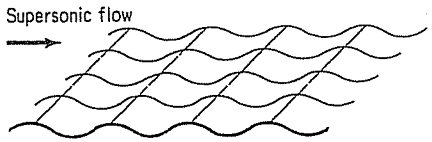

Figure 11.13 A wavy plate in a steady supersonic flow.

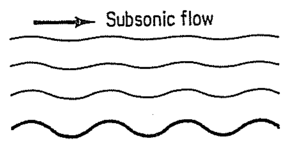

Figure 11.14 A wavy plate in a steady subsonic flow.

The amplitude a is assumed to be small compared with the wave length L:

$$a \ll L. \tag{11.12–12}$$

The fluid, since it is assumed to be perfect, can glide over the plate, but cannot penetrate it. Therefore, the velocity vector of the flow must be tangent to the plate. Now the velocity vector has the components

$$U + u', v', w' \tag{11.12–13}$$

in the x-, y-, and z-directions, respectively. On the other hand, the normal vector to the surface described by Eq. (11.12–11) has the following components (see Fig. 11.15):

$$-\frac{\partial z}{\partial x}, \quad -\frac{\partial z}{\partial y}, \quad 1. \tag{11.12–14}$$

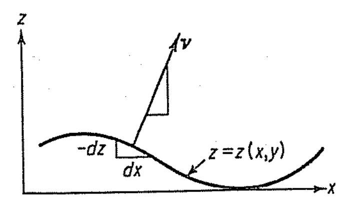

Figure 11.15 Surface normal and the velocity boundary condition.

If the velocity vector, with components given by Eq. (11.12–13), is to be tangent to the surface of the plate, it must be normal to the normal vector given by Eq. (11.12–14). Hence, the condition of nonpenetration can be stated as the orthogonality of the vectors of Eqs. (11.12–13) and (11.12–14), i.e., by the condition that their scalar product vanishes:

$$-(U + u')\frac{\partial z}{\partial x} - v'\frac{\partial z}{\partial y} + w' \cdot 1 = 0.$$

Omitting higher-order terms, we obtain the boundary condition

$$w' = U\frac{\partial z}{\partial x} \tag{11.12–15}$$

on the plate. From Eq. (11.12–11), this is

$$w' = U\frac{a\pi}{L}\cos\frac{\pi x}{L} \quad \left(\text{when } z = a\sin\frac{\pi z}{L}\right). \tag{11.12–16}$$

Again, counting on the continuity and differentiability of the function $w'(x, y, z)$, we can write

$$w'(x, y, z) = w'(x, y, 0) + z\left(\frac{\partial w'}{\partial z}\right)_{z=0} + \dots. \tag{11.12–17}$$

For small z, all the terms following the first are higher-order terms. Consistently neglecting these terms, we can simplify the boundary condition to

$$w' = U\frac{a\pi}{L}\cos\frac{\pi x}{L} \quad (\text{when } z = 0). \tag{11.12–18}$$

Condition at Infinity

The boundary condition given in Eq. (11.12–18) is not sufficient to determine the solution to our problem, which is governed by either Eq. (11.12–9) or Eq. (11.12–10), depending on whether the flow is subsonic or supersonic. In addition, the conditions at infinity must be specified. There is a great difference between the elliptic and hyperbolic equations with respect to the appropriate types of boundary conditions that may be specified, and we must consider them in some detail.

Subsonic case. For the elliptic equation (11.12–9), the influence of the disturbances is spread out in all directions, and it is reasonable to assume that, for any finite body, the disturbances tend toward zero at distances infinitely far away from the body. A rigorous argument may be based on the total energy that may be imparted to the fluid. If the fluid velocity is distributed in a certain fashion, and if it does not tend toward zero at a certain rate as the distance from the body increases toward infinity, an infinitely large energy would have to be imparted to the fluid in order to create the motion, which is impossible. (For further details, see texts on partial differential equations or aerodynamics.) Accordingly, we impose the following conditions on our problem:

(a) The flow is two-dimensional and parallel to the xz-plane, and there is no dependence on the y-coordinate.

(b) All disturbances tend toward zero as $z \to \pm\infty$. In particular,

$$u', v', w' \to 0; \quad \text{i.e.,} \quad \Phi \to \text{const. as } z \to \pm\infty. \tag{11.12–19}$$

Supersonic case. Turning now to the hyperbolic equation (11.12–10), we find that the disturbances can be carried away along waves of limited dimension. The argument of decreasing amplitude does not apply. Instead, the boundary condition must be replaced by the *radiating condition*: that the plate is the only source of disturbances and that the disturbances radiate *from* the source, not toward it.

This description of the radiation condition is easy to apply when we are concerned with a single source. For example, of the two solutions on the right-hand side of Eq. (11.11–13), the term $f(r - ct)/r$ represents a wave radiating from the origin; hence, for a source at the origin, it is the only term admissible under the radiation condition. The condition becomes somewhat confounded, however, when applied to two-dimensional steady flow. Perhaps the matter can be clarified by examining some photographs of supersonic flow about stationary models in a wind tunnel, such as those shown in Fig. 11.16. Here, the flows are from left to right. We see that the lines of disturbances, which are contours of density of the fluid as revealed by the Schlieren photographs, incline to the right. This direction of inclination of the strong (shock) and weak (Mach) waves is determined by the radiation condition.

Solution of the Wavy Wall Problem

Now we can return to our problem. It is easily verified by direct substitution that, in the subsonic case, Eq. (11.12–9) can be satisfied by a function of the form

$$\Phi = Ae^{\mu z} \cos \frac{\pi x}{L}. \tag{11.12–20}$$

On substituting Eq. (11.12–20) into Eq. (11.12–9), we obtain

$$-(1 - M^2)\left(\frac{\pi}{L}\right)^2 Ae^{\mu z} \cos \frac{\pi x}{L} + A\mu^2 e^{\mu z} \cos \frac{\pi x}{L} = 0,$$

or

$$\mu = \pm\left(\frac{\pi}{L}\right) \sqrt{1 - M^2}. \tag{11.12–21}$$

If the plus sign is used in Eq. (11.12–21), the function Φ in Eq. (11.12–20) will grow exponentially without limit as $z \to \infty$. On the other hand, if the minus sign is used, Eq. (11.12–19) can be satisfied. Hence, we may try

$$\Phi = Ae^{-(\pi/L)\sqrt{1-M^2}z} \cos \frac{\pi x}{L}. \tag{11.12–22}$$

The vertical velocity w' computed from Φ is

$$w' = \frac{\partial \Phi}{\partial z} = -\frac{\pi}{L}\sqrt{1 - M^2}\, Ae^{-(\pi/L)\sqrt{1-M^2}z} \cos \frac{\pi x}{L}. \tag{11.12–23}$$

On setting $z = 0$ in Eq. (11.12–23) and applying the boundary condition, Eq. (11.12–18), we obtain

$$A = -\frac{U a}{\sqrt{1 - M^2}}. \tag{11.12–24}$$

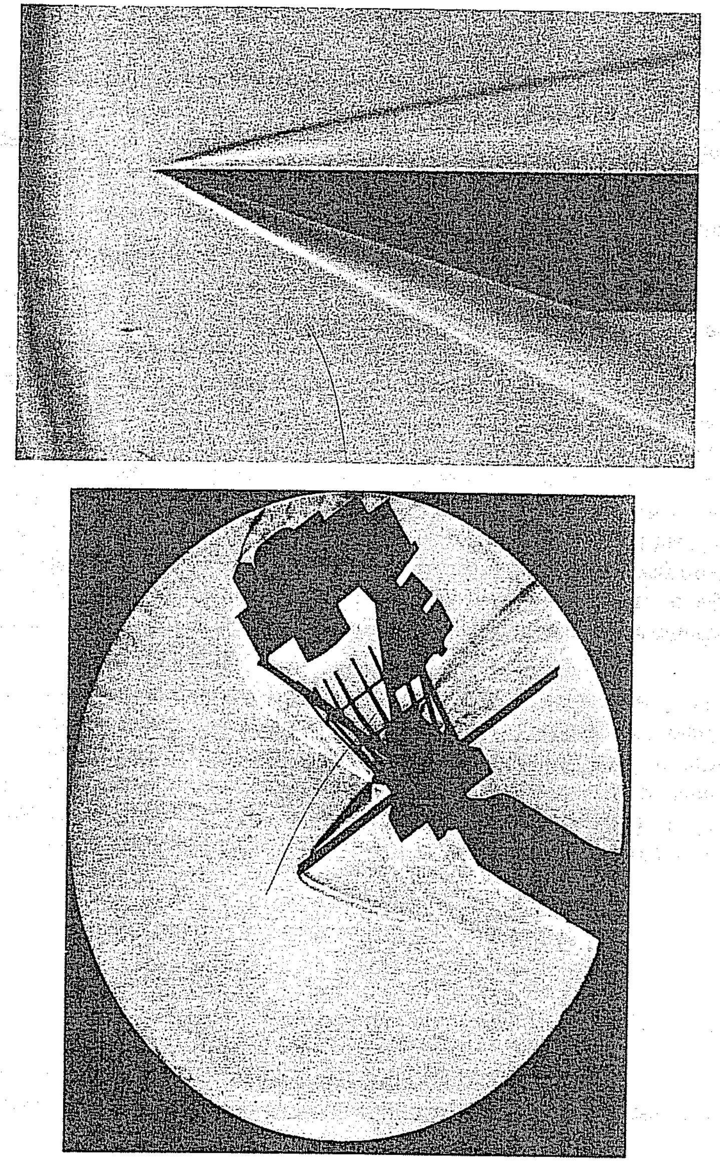

Figure 11.16 (a) Flow past a flat plate with a beveled, sharp leading edge, the top surface being aligned with the free stream of Mach number 8. On the top side of the plate, a laminar boundary layer is revealed by the lighter line. A shock wave is induced by the displacement effect of the boundary layer. Similar features are seen on the lower side. Schlieren system. Flow left to right. *Courtesy of Toshi Kubota, California Institute of Technology*; (b) Scale mode of the Nimbus spacecraft in a 50-in hypersonic tunnel, at Mach number 8 and Reynolds number of 0.42 × 10^6/ft. Schlieren system. Flow left to right. *Courtesy of Von Karman Gas Dynamics Facility, ARO, Inc.*

Now all the boundary conditions for the subsonic case are satisfied. Hence, the solution for the subsonic case is

$$\Phi = -\frac{U\,a}{\sqrt{1 - M^2}}\, e^{-(\pi/L)\sqrt{1-M^2}z} \cos\frac{\pi x}{L}. \qquad (11.12\text{–}25)$$

We see that the disturbances decrease exponentially with increasing z. From this solution, we can deduce the velocity field, the pressure field, and the density field. In particular, since

$$U\frac{\partial u'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial x'}, \qquad (11.12\text{–}26)$$

we have

$$p' = -\rho U u' = -\rho U \frac{\partial \Phi}{\partial x}. \qquad (11.12\text{–}27)$$

The streamlines for such a flow are plotted in Fig. 11.14.

Turning now to the supersonic case, Eq. (11.12–10), we see that it can be satisfied by the function

$$\Phi = f(x - \sqrt{M^2 - 1}\,z) + g(x + \sqrt{M^2 - 1}\,z), \qquad (11.12\text{–}28)$$

where f and g are arbitrary functions, because if we set

$$\xi = x - \sqrt{M^2 - 1}\,z, \qquad (11.12\text{–}29)$$

then

$$\frac{\partial f}{\partial x} = \frac{df}{d\xi}, \qquad \frac{\partial f}{\partial z} = -\sqrt{M^2 - 1}\,\frac{df}{d\xi};$$

hence,

$$(M^2 - 1)\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial z^2} = (M^2 - 1)\frac{d^2 f}{d\xi^2} - (M^2 - 1)\frac{d^2 f}{d\xi^2} = 0,$$

and Eq. (11.12–10) is satisfied. The lines

$$\xi = \text{const.}, \quad \text{i.e.,} \quad x - \sqrt{M^2 - 1}\,z = \text{const.}, \qquad (11.12\text{–}30)$$

are the Mach waves, along which the disturbances are propagated with undiminished intensity. These lines are inclined in the correct direction, as revealed by the wind-tunnel photographs. On the other hand, the Mach lines for the function $g(x + \sqrt{M^2 - 1}\,z)$ are inclined in the wrong direction. Hence, the function g must be rejected on the basis of the radiation condition. Therefore, we may try

$$\Phi = f(x - \sqrt{M^2 - 1}\,z). \qquad (11.12\text{–}31)$$

From Eq. (11.12–31), we obtain

$$w' = \frac{\partial \Phi}{\partial z} = -\sqrt{M^2 - 1}\,\frac{df}{d\xi}. \qquad (11.12\text{–}32)$$

Comparing Eq. (11.12–32) with the boundary condition, Eq. (11.12–18), we obtain, when $z = 0$,

$$-\sqrt{M^2 - 1}\left(\frac{df}{d\xi}\right)_{z=0} = \frac{U\,a\pi}{L}\cos\frac{\pi x}{L} = \frac{U\,a\pi}{L}\cos\frac{\pi\xi}{L}\bigg|_{z=0}. \qquad (11.12\text{–}33)$$

Hence, on integrating and returning to Eq. (11.12–29), we have

$$\Phi = f = -\frac{U\,a}{\sqrt{M^2 - 1}}\sin\frac{\pi}{L}(x - \sqrt{M^2 - 1}\,z), \qquad (11.12\text{–}34)$$

which solves the problem. A plot of the streamlines is shown in Fig. 11.13.

The contrast between the two cases is dramatic. Whereas in the subsonic case the pressure disturbance is diminished as the distance from the plate increases, in the supersonic case it is not. This is, of course, the reason why a sonic boom hits us with all its fury from a supersonic aircraft, but not from a subsonic one.

11.13 APPLICATIONS TO BIOLOGY

Fluid mechanics is as relevant to living creatures as to machines and physical objects. The gas in the airway and lung, the urine, and the sap in the xylem of trees are Newtonian fluids to which the Navier-Stokes equation and the no-slip boundary conditions apply. Blood is a non-Newtonian fluid. If the shear strain rate is sufficiently high (e.g., > 100 s^{-1}), the viscosity of blood is almost constant, i.e., its behavior is almost Newtonian. If the shear strain rate is low, however, the viscosity of blood increases. Saliva, mucus, synovial fluid in the knee joint, and other body fluids are also non-Newtonian. Analysis of the flow of these must take their non-Newtonian behavior into consideration.

Blood can be treated as a homogeneous fluid only when one is considering flow in a blood vessel whose diameter is much larger than the diameter of the red blood cells. Flow in a small blood vessel, such as in the capillaries, whose diameter is about the same as that of the red cells, must treat the cells as individual bodies. The blood is, then, a biphasic fluid. Other body fluids that contain proteins and other suspensions may have to be treated as biphasic or multiphasic if the dimensions of the vessels in which they flow are sufficiently small.

Animals and plants live in gas, water, and earth. Understanding their movements requires fluid mechanics. Body fluids circulate inside of animals and plants. Understanding their movement also requires fluid mechanics. In either case, the boundary conditions are, in general, nonstationary.

The examples considered in this chapter have applications to biology. The analysis of the flow in a channel or tube is relevant to the blood flow problem. The blood vessels, however, are elastic. The diameter of a blood vessel varies with the pressure. The interaction between the flow and the elastic deformation of the wall can produce some very interesting phenomena. In biology, solid mechanics and fluid mechanics are often closely knit together.

The reader may gain some insight into the broad subject of fluid mechanics and biomechanics from the references listed at the end of the chapter.

PROBLEMS

11.9 Derive the Navier-Stokes equation for an incompressible fluid in cylindrical polar coordinates.

Solution: The left-hand side of the Navier-Stokes equation represents acceleration. In polar coordinates, the components are a_r, a_θ, a_z, which are given in Eq. (10.9–9) on p. 225. The right-hand side is the vector divergence of the stress tensor. In polar coordinates, these components are given by Eq. (10.9–11). It remains to write down the stresses in terms of the velocities u, v, w along the radial, circumferential, and axial directions, respectively. On p. 128, we have e_{rr}, $e_{\theta\theta}$, etc., expressed in terms of u_r, u_θ, u_z. The relationship between the strain rates $\dot{e}_{rr}$, $\dot{e}_{\theta\theta}$, etc., to the velocities u, v, w are the same. Hence,

$$\dot{e}_{rr} = \frac{\partial u}{\partial r}, \qquad \dot{e}_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \text{ etc.}$$

Therefore, from Eq. (7.3–6), and for an incompressible fluid, we have

$$\sigma_{rr} = -p + 2\mu\dot{e}_{rr} = -p + 2\mu\frac{\partial u}{\partial r},$$

$$\sigma_{\theta\theta} = -p + 2\mu\dot{e}_{\theta\theta} = -p + 2\mu\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right),$$

$$\sigma_{zz} = -p + 2\mu\dot{e}_{zz} = -p + 2\mu\frac{\partial w}{\partial z},$$

$$\sigma_{r\theta} = 2\mu\dot{e}_{r\theta} = \mu\left(r\frac{\partial(v/r)}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right),$$

$$\sigma_{\theta z} = 2\mu\dot{e}_{\theta z} = \mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}\right),$$

$$\sigma_{zr} = 2\mu\dot{e}_{zr} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right).$$

A substitution into Eq. (10.9–11) yields the Navier-Stokes equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) + F_r$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right) + F_\theta$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w + F_z$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0.$$

11.10 Blood is a non-Newtonian fluid whose viscosity varies with the strain rate. (See Fig. 9.15 and Prob. 9.4.) Derive the equation of motion of blood in a form analogous to the Navier-Stokes equations. Formulate mathematically the problem of blood flow in a living heart.

11.11 If air is truly nonviscous, would an airplane be able to fly? What about birds and insects? Why?

11.12 If water is nonviscous, would fish be able to swim? What are the differences in the arguments for fish in water and birds in the air?

11.13 Formulate the mathematical problem of tides induced on the earth under the influence of the moon. (See Lamb, *Hydrodynamics*, pp. 358–362.)

11.14 Waves are generated in water in a long channel of rectangular cross section. What are the equations with which the wavelength and frequency can be determined?

11.15 Ripples are generated on the surface of water in a deep pond. Does the wave speed depend on the wavelength? Even though the full solution is rather complicated, whether or not the waves are dispersive (i.e., whether the speed depends on wavelength) can be detected when all the basic equations are written down. Take the free surface of the pond to be the xy-plane, let the z-axis point downward, and try a two-dimensional solution with velocity components

$$v \equiv 0, \qquad u = ae^{-kz}\sin kx \sin \omega t, \quad \text{and} \quad w = -ae^{-kz}\cos kx \sin \omega t.$$

11.16 Consider a ground-effect machine, which uses one or more reaction jets and hovers above the ground. Sketch the streamlines of the flow and write the equations and boundary conditions that govern the machine when it is hovering.

11.17 Analyze the motion in a cumulus cloud in a summer thunderstorm. What are the variables relevant to this problem? If temperature is an important consideration, how would it be incorporated into the basic equations? Gravity must not be neglected. Present the basic equations. Make a dimensional analysis to determine fundamental dimensionless parameters.

11.18 Water waves run up a sloping beach and create all the panorama on the seashore: surf, riptides, waves, ripples, and foam. Analyze the phenomenon mathematically. Give an appropriate choice of variables. Write down the differential equations and boundary conditions. Make simplifying assumptions if you think they are appropriate, but state your assumptions clearly.

11.19 On the beach there are riptides, which are fast-moving narrow streams of water that move toward the ocean in a direction perpendicular to the shoreline and are dangerous to swimmers. Now this is an anomaly: For a two-dimensional sloping beach and a two-dimensional water wave, we obtain a three-dimensional solution. Is there any basic objection to this situation (from the mathematician's point of view, not the swimmer's)? Can you name another example of such a phenomenon in nature?

11.20 When wind blows over (perpendicular to) long cylindrical pipes, vortices are shed in the wake. These vortices induce vibrations in the pipe. A trans-Arabic oil line (the aboveground part) was reported to have suffered severe vibrations due to wind. Smokestacks, large rockets, and the like are subjected to these disturbances. Vortex shedding over a long cylinder is three dimensional; in other words, the shedding is nonuniform along the length of the cylinder, even if the wind and the cylinder are both uniform. Formulate the aerodynamic problem for a fixed, rigid cylinder. Furnish all the differential equations and boundary conditions. Make a dimensional analysis to determine all the dimensionless parameters involved.

11.21 Generalize Prob. 11.20 to take account of the vortex shedding over a flexible, vibrating cylinder.

11.22 Using the equations derived in Prob. 11.9, find the velocity field in a Couette flowmeter (Fig. P3.22, p. 86).

Answer: Let $v = \omega_1 a$ at $r = a$ and $v = \omega_2 b$ at $r = b$. Then

$$v = (a^2 - b^2)^{-1}[(\omega_1 a^2 - \omega_2 b^2)r - a^2 b^2(\omega_2 - \omega_1)/r].$$

11.23 Using the Navier-Stokes equation, find the velocity distribution of a flow in a long cylindrical pipe of rectangular cross section.

11.24 Discuss whether the concept of a boundary layer is important in each of the following problems. Explain briefly how and why boundary-layer theory is used in those problems to which it is applicable.

(a) Blood flow in the aorta. Assume a viscosity coefficient $\mu = 0.04$ poise, radius $r = 3$ mm, density $\rho = 1$, and velocity $v = 50$ cm/sec.

(b) Blood flow in small blood vessels. Assume a coefficient of viscosity $\mu = 0.04$ poise, radius $a = 10^{-3}$ cm, density $\rho = 1$, and mean velocity $v = 0.07$ cm/sec.

Note: Compute the Reynolds number $R_N = 2\,VL/\mu$. In (a), $R_N = 750$. In (b), $R_N = 3.5 \times 10^{-3}$. The boundary layer thickness δ is on the order of $(R_N)^{-1/2}$.

11.25 I have a garden hose curved on the ground. One end is connected to a water faucet. When the valve is opened, the pressure is high, a water jet comes out with good force, and the hose whips like a snake. Why?

Now consider an analagous problem for a pipeline suspended in air above ground. One span L is supported between two pillars. The pipe is a thin-walled circular cylindrical shell, in which flows a fluid. The pipe is straight if there is no load. It is loaded by its own weight, the weight of the fluid, and the pressure of the flowing fluid. To design the pipe and the pillars, what fluid mechanical problems should be considered? Formulate a mathematical theory for an important problem that you identified. Write down the differential equations and the boundary conditions. Outline a method of solution.

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12 SOME SIMPLE PROBLEMS IN ELASTICITY

Basic equations, elastic waves, torsion of shafts, bending of beams, and some remarks about biomechanics.

12.1 BASIC EQUATIONS OF ELASTICITY FOR HOMOGENEOUS, ISOTROPIC BODIES

In the preceding chapter, we discussed the equations governing the flow of fluids. In this chapter, we shall consider the motion of solids that obey the Hooke's law. A Hookean body has a unique zero-stress state. All strains and particle displacements are measured from this state, in which their values are counted as zero.

The basic equations can be gleaned from the preceding chapters. Let $u_i(x_1, x_2, x_3, t)$, $i = 1, 2, 3$, describe the displacement of a particle located at x_1, x_2, x_3 at time t from its position in the zero-stress state. Various strain measures can be defined for the displacement field. The Green strain tensor is expressed in terms of $u_i(x_1, x_2, x_3, t)$ according to Eq. (5.3–3):

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]. \tag{12.1–1}$$

Here, and hereinafter, all Latin indices range over 1, 2, 3. The particle velocity v_i is given by the material derivative of the displacement,

$$v_i = \frac{\partial u_i}{\partial t} + v_j\frac{\partial u_i}{\partial x_j}. \tag{12.1–2}$$

The particle acceleration α_i is given by the material derivative of the velocity, Eq. (10.3–7),

$$\alpha_i = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}. \tag{12.1–3}$$

The conservation of mass is expressed by the equation of continuity, Eq. (10.5–3),

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0. \tag{12.1–4}$$

The conservation of momentum is expressed by the Eulerian equation of motion, Eq. (10.6–7),

$$\rho\alpha_i = \frac{\partial \sigma_{ij}}{\partial x_j} + X_i. \tag{12.1–5}$$

Hooke's law for a homogeneous, isotropic material is

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2Ge_{ij}, \tag{12.1–6}$$

where λ and G are Lamé constants.

Equations (12.1–1) through (12.1–6) together describe a theory of elasticity. If we compare these equations with the corresponding equations for a viscous fluid, as given in Sec. 11.1, we see that their theoretical structures are similar, except that here we have a nonlinear strain-and-displacement-gradient relation [Eq. (12.1–1)], in contrast to the linear rate-of-deformation-and-velocity-gradient relation [Eq. (6.1–3)], for the fluid. Hence, the theory of elasticity is more deeply nonlinear than the theory of viscous fluids.

The nonlinear problem is so wrought with mathematical complexities that only a few exact solutions are known. For this reason, it is common to simplify the theory by introducing a severe restriction, namely, that *the displacements and velocities are infinitesimal.* In this way, Eqs. (12.1–1) through (12.1–3) can be linearized. One tries to learn as much as possible about the linearized theory and then proceed to discover what features are introduced by the nonlinearities.

We *linearize* the equations by restricting ourselves to values of u_i, v_i so small that the nonlinear terms in Eqs. (12.1–1) through (12.1–3) may be neglected. Thus,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{12.1–7}$$

$$v_i = \frac{\partial u_i}{\partial t}, \qquad \alpha_i = \frac{\partial v_i}{\partial t}. \tag{12.1–8}$$

Equations (12.1–4) through (12.1–8) together are 22 equations for the 22 unknowns $\rho, u_i, v_i, \alpha_i, e_{ij}, \sigma_{ij}$; $i, j = 1, 2, 3$. We may eliminate σ_{ij} by substituting Eq. (12.1–6) into Eq. (12.1–5) and using Eq. (12.1–7) to obtain the well-known *Navier's equation,*

$$G\nabla^2 u_i + (\lambda + G)\frac{\partial e}{\partial x_i} + X_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \tag{12.1–9}$$

where e is the divergence of the displacement vector $\mathbf{u}$; i.e.,

$$e = \frac{\partial u_j}{\partial x_j} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{12.1–10}$$

∇^2 is the *Laplace operator*. If we write x, y, z instead of x_1, x_2, x_3, we have

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{12.1–11}$$

If we introduce Poisson's ratio ν, as in Eq. (7.4–8), we can write Navier's equation (12.1–9) as

$$G\left(\nabla^2 u_i + \frac{1}{1-2\nu}\frac{\partial e}{\partial x_i}\right) + X_i = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{12.1–12}$$

This is the basic field equation of the linearized theory of elasticity.

Navier's equation (12.1–9) must be solved for appropriate boundary conditions, which are usually one of two kinds:

(1) *Specified displacements*. The components of displacement u_i are prescribed on the boundary.

(2) *Specified surface tractions*. The components of surface traction $\overset{\nu}{T}_i$ are assigned on the boundary.

In most problems of elasticity, the boundary conditions are such that over one part of the boundary the displacements are specified, whereas over another part the surface tractions are specified. In the latter case, Hooke's law may be used to convert the boundary condition into prescribed values of a certain combination of the first derivatives of u_i.

12.2 PLANE ELASTIC WAVES

To illustrate the use of the linearized equations, let us consider a simple harmonic wave train in an elastic medium. Let us assume that the displacement components u_1, u_2, u_3 (or, in unabridged notation, u, v, w) are infinitesimal and that the body force X_i vanishes. Then it is easy to verify that a solution of Navier's equation (12.1–9) is

$$u = A\sin\frac{2\pi}{l}(x \pm c_L t), \qquad v = w = 0, \tag{12.2–1}$$

where A, l, and c_L are constants, provided that the constant c_L is chosen to be

$$c_L = \sqrt{\frac{\lambda + 2G}{\rho}} = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}. \tag{12.2–2}$$

The pattern of motion expressed by Eq. (12.2–1) is unchanged when $x \pm c_L t$ remains constant. Hence, if the negative sign were taken, the pattern would move to the

right with a velocity c_L as the time t increased. The constant c_L is called the *phase velocity* of the wave motion. The constant l is the *wavelength*, as can be seen from the sinusoidal pattern of u as a function of x, at any instant of time. The particle velocity computed from Eq. (12.2–1) is in the same direction as that of the wave propagation (namely, in the direction of the x-axis). Such a motion is said to constitute a train of *longitudinal waves*. Since at any instant of time the wave crests lie in parallel planes, the motion represented by this equation is called a train of *plane waves*.

Next, let us consider the motion

$$u = 0, \qquad v = A\sin\frac{2\pi}{l}(x \pm ct), \qquad w = 0, \tag{12.2–3}$$

which represents a train of plane waves of wavelength l propagating in the direction of the x-axis with a phase velocity c. When Eqs. (12.2–3) are substituted into Eq. (12.1–9), it is seen that c must assume the value

$$c_T = \sqrt{\frac{G}{\rho}}. \tag{12.2–4}$$

The particle velocity (in the y-direction) computed from Eq. (12.2–3) is perpendicular to the direction of wave propagation (the x-direction). Hence, the wave generated is said to be a *transverse wave*. The speeds c_L and c_T are called the characteristic *longitudinal wave speed* and *transverse wave speed*, respectively. They depend on the elastic constants and the density of the material. The ratio c_T/c_L depends on Poisson's ratio only and is given by

$$c_T = c_L\sqrt{\frac{1-2\nu}{2(1-\nu)}}. \tag{12.2–5}$$

If $\nu = 0.25$, then $c_L = \sqrt{3}\, c_T$.

Similar to Eq. (12.2–3), the following equations represent a transverse wave in which the particles move in the direction of the z-axis:

$$u = 0, \qquad v = 0, \qquad w = A\sin\frac{2\pi}{l}(x \pm c_T t). \tag{12.2–6}$$

The plane parallel to which the particles move [such as the xy-plane in Eq. (12.2–3) or the xz-plane in Eq. (12.2–6)] is called the *plane of polarization*.

Plane waves may exist only in an unbounded elastic continuum. In a finite body, a plane wave will be reflected when it hits a boundary. If there is another elastic medium beyond the boundary, refracted waves occur in the second medium. The features of reflection and refraction are similar to those in acoustics and optics; the main difference is that, in elasticity, an incident longitudinal wave will be reflected and refracted in a combination of longitudinal and transverse waves, and an incident transverse wave will also be reflected in a combination of both types of waves. The details can be worked out by the proper combination of these waves so that the boundary conditions are satisfied.

12.3 SIMPLIFICATIONS

Important simplifications to the equation of the linearized theory of elasticity may come from

(1) Homogeneity and isotropy.
(2) The absence of inertial forces.
(3) A high degree of symmetry in geometry.
(4) Plane stress and plane strain.
(5) Thin-walled structures—plates and shells.

Clearly, a simplification is obtained if the number of independent or dependent variables is reduced. Thus, if nothing changes with time, the variable t will be suppressed. Homogeneity of materials makes the coefficients of the differential equations constant. Isotropy reduces the number of independent material constants. High degree of symmetry reduces the number of geometric parameters in a problem. Reduction of the general field equations to two dimensions or one dimension reduces the number of independent and dependent variables.

Example 1. A Plane State of Stress

A *plane-stress* state depending on x, y only may be visualized as a state that exists in a thin membrane stressed in its own plane. Figure 4.1 on p. 89 shows an example of such a case. Analytically, a plane-stress state is defined by the condition that the stress components σ_{zz}, σ_{zx}, σ_{zy} vanish everywhere, i.e.,

$$\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0, \qquad (12.3\text{–}1)$$

whereas the stress components σ_{xx}, σ_{xy}, σ_{yy} are independent of the coordinate z.

Example 2. A Plane State of Strain

If the z-component of the displacement w vanishes everywhere, and if the displacements u, v are functions of x, y only, and not of z, the body is said to be in a *plane-strain* state, depending on x, y only. Such a state may be visualized as one that exists in a long cylindrical body loaded uniformly along the axis. With a plane-strain state, we must have

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0. \qquad (12.3\text{–}2)$$

12.4 TORSION OF A CIRCULAR CYLINDRICAL SHAFT

We shall now illustrate an application of the linearized elasticity theory by considering the problem of torsion. To transmit a torque from one place to another, a shaft is employed. The problem is to solve the Navier's equations to obtain the stress distribution in the shaft. The degree of difficulty to solve this problem depends on the geometry of the shaft. If the shaft is a circular cylinder, the solution is simple. If it is a cylinder of a noncircular cross section, or if the shaft has variable cross sections, then it is difficult.

Let us consider the simple problem of the torsion of a cylindrical shaft of circular cross section. (See Fig. 12.1, which shows the notations and the coordinate axes to be used.) Before tackling the problem analytically, let us look at the physical conditions. Under the torque, the shaft twists. Let the cross section at $z = 0$ be fixed. Since the shaft, as well as the loading, is homogeneous along the z-axis, the twist must be uniform along the z-axis. Hence, the deformation must be expressible in terms of twist per unit length α, which is a constant independent of z. The quantity α represents the rotation of a section at $z = 1$ relative to that at $z = 0$.

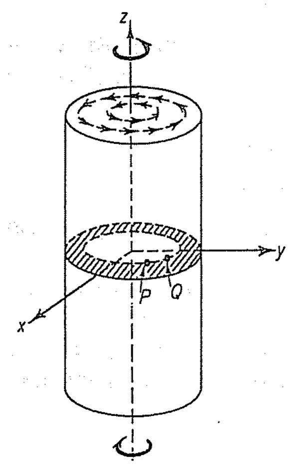

Figure 12.1 Torsion of a circular shaft.

By reason of symmetry, it is obvious that a circular cross section of the shaft remains circular when a torque is applied. But what about the axial displacements of such a section? Consider a plane cross section such as that at $z = 0$ before the torque is applied. When the torque T is applied, the boundary conditions are axisymmetric, hence any axial displacement must be axisymmetric. But the boundary conditions are also preserved by reversal of the z-axis, hence the axial displacement must be zero, and the plane section remains plane.

Summarizing this discussion, we see that the distortion of a circular shaft under a torque must be a relative rotation of the cross sections at a uniform rate of twist. Therefore, the displacement of a particle located at (x, y, z) would appear to be, in polar coordinates,

$$u_r = 0, \qquad u_\theta = \alpha z r, \qquad u_z = 0, \qquad (12.4\text{–}1)$$

or, in rectangular Cartesian coordinates,

$$u_x = -\alpha zy, \qquad u_y = \alpha zx, \qquad u_z = 0, \tag{12.4-2}$$

as is shown in Fig. 12.2.

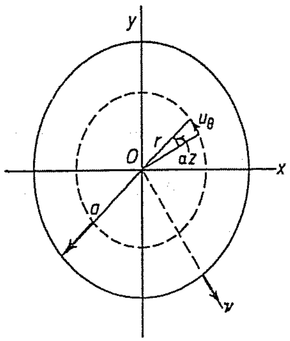

Figure 12.2 Notations.

We shall now show that this is indeed correct. Since the displacements are given, there is no need to check the compatibility conditions. We must, however, check the equation of equilibrium and the boundary conditions.

From Eq. (12.4-2), we have the strain components

$$e_{xx} = 0, \qquad e_{yy} = 0, \qquad e_{zz} = 0,$$

$$\begin{aligned} e_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}(\alpha z - \alpha z) = 0, \\ e_{xz} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = -\frac{1}{2}\alpha y, \\ e_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2}\alpha x. \end{aligned} \tag{12.4-3}$$

The stress-strain relation yields the corresponding stress components. We have

$$\begin{aligned} &\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0, \\ &\sigma_{xz} = -G\alpha y, \\ &\sigma_{yz} = G\alpha x, \end{aligned} \tag{12.4-4}$$

where G is the shear modulus of the shaft material.

The equations of equilibrium are obtained by omitting α_i and X_i in Eq. (12.1-5). We obtain

$$\begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \\ &\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0, \end{aligned} \tag{12.4-5}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0,$$

which are obviously satisfied by the stress components given in Eq. (12.4-4).

The boundary conditions of our problem consists of the facts that the lateral surfaces are stress free and that the ends are acted on by a torque. Since there is no tension or compression on the ends, we have

$$\sigma_{zz} = 0 \qquad (\text{on } z = -L \text{ and } z = L). \tag{12.4-6}$$

This is satisfied by Eq. (12.4-4).

The stress vector acting on the lateral surface is given by $\overset{\nu}{T}_i$, where $\boldsymbol{\nu}$ denotes the vector normal to the lateral surface. By Cauchy's formula,

$$\overset{\nu}{T}_i = \nu_j \sigma_{ij}. \tag{12.4-7}$$

Setting $i = 1, 2, 3$, we have the three equations,

$$\begin{aligned} &\sigma_{xx}\nu_x + \sigma_{xy}\nu_y + \sigma_{xz}\nu_z = 0, \\ &\sigma_{yx}\nu_x + \sigma_{yy}\nu_y + \sigma_{yz}\nu_z = 0, \\ &\sigma_{zx}\nu_x + \sigma_{zy}\nu_y + \sigma_{zz}\nu_z = 0, \end{aligned} \tag{12.4-8}$$

where ν_x, ν_y, ν_z are the direction cosines of the normal vector to the lateral surface. Now, on the lateral surface, it is evident from Fig. 12.2 that the normal vector $\boldsymbol{\nu}$ coincides with the radius vector. Hence, the components of $\boldsymbol{\nu}$ are

$$\nu_x = \frac{x}{a}, \qquad \nu_y = \frac{y}{a}, \qquad \nu_z = 0. \tag{12.4-9}$$

Consequently, the boundary conditions on the circumference C are

$$\begin{aligned} &x\sigma_{xx} + y\sigma_{xy} = 0, \\ &x\sigma_{yx} + y\sigma_{yy} = 0, \\ &x\sigma_{zx} + y\sigma_{zy} = 0, \end{aligned} \tag{12.4-10}$$

which are again satisfied by Eq. (12.4-4).

It remains to check the condition that the stresses acting on the ends $z = -L$ and $z = L$ are equipollent to a torque. Referring to Fig. 12.3 and using Eq.

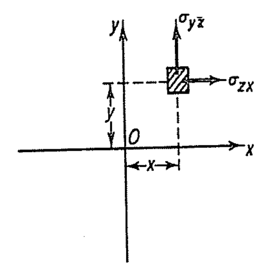

Figure 12.3 Stresses in a twisted shaft.

(12.4–4), we see that the resultant of the stresses acting on the end cross sections are

$$\iint \sigma_{xz}\,dx\,dy = -G\alpha \iint y\,dx\,dy = -G\alpha \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y\,dy = 0,$$

$$\iint \sigma_{yz}\,dx\,dy = G\alpha \iint x\,dx\,dy = 0, \tag{12.4–11}$$

$$\iint \sigma_{zz}\,dx\,dy = 0.$$

Hence, the resultant force vanishes as desired. The resultant moment about the z-axis is, however,

$$\iint (x\sigma_{yz} - y\sigma_{xz})dx\,dy. \tag{12.4–12}$$

On substituting from Eq. (12.4–4), we have

$$\text{moment} = G\alpha \iint (x^2 + y^2)\,dx\,dy$$
$$= G\alpha \int_0^{2\pi} d\theta \int_0^a r^3\,dr$$
$$= \frac{2\pi G\alpha a^4}{4}.$$

Thus, we see that the resultant moment is indeed a torque of magnitude T:

$$T = \frac{\pi a^4 G\alpha}{2} \tag{12.4–13}$$

The checking is now complete. All the equations of equilibrium and the boundary conditions are satisfied. The solution contained in Eqs. (12.4–1) through (12.4–4) is exact.

PROBLEM

12.1 Consider the torsion of a shaft of square cross section. Write down all the boundary conditions. Show that the solution contained in Eqs. (12.4–1) through (12.4–4) no longer satisfy all the boundary conditions.

12.5 BEAMS

When a structural member is used to transmit bending moment and transverse shear, it is called a *beam*. Beams are used constantly in engineering and, therefore, are important objects for study. The floor we stand on is resting on beams. An airplane wing is a beam. Bridges are made of beams, and so on. An engineer should know the stress and deformation in a beam, how to choose the materials

for a beam, how to use the material efficiently by a proper geometric design, how to minimize the beam's weight, how to maximize the stiffness and stability of the beam, how to utilize supports to minimize vibrations, how to calculate the loads that act on the beam (static and moving loads, wind loads on a building, aerodynamic load on an airplane, etc.), how to analyze aeroelastic or hydroelastic interactions in case a beam is used in a fluid flow (such as with an airplane wing, or the structure of a ship), and more.

Beams are classified according to the condition of support at their ends. An end is called *simply supported* when it is free to rotate, but is restrained from lateral translation. An end is said to be *free* when it is free to rotate and deflect. An end is said to be *clamped* when translation and rotation are both prevented.

In Sec. 1.11, we considered the pure bending of a prismatic beam of a homogeneous, isotropic Hookean material. We deduced certain results, but we did not check all the field and boundary conditions. We shall now show that all these conditions are satisfied.

Consider the pure bending of a prismatic beam, as shown in Fig. 1.14 on p. 26. Let the beam be subjected to two equal and opposite couples M acting in a plane of symmetry of the cross sections of the beam. Let the x, y, z-axes of reference be chosen as in Sec. 7.7, with the origin located at the centroid of a cross section. In Sec. 7.7, we were led to the conclusion that the stress distribution in the beam is

$$\sigma_{xx} = \frac{Ey}{R}, \qquad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0, \tag{12.5–1}$$

$$\frac{M}{EI} = \frac{1}{R}, \qquad \sigma_{xx} = \sigma_o\frac{y}{c}, \qquad \sigma_o = \frac{Mc}{I}, \tag{12.5–2}$$

where c is the distance from the neutral surface to the "outer fiber" of the cross section, M is the bending moment, E is Young's modulus, I is the areal moment of inertia of the cross section, and σ_o is the outer fiber stress. The strains are, therefore,

$$e_{xx} = \frac{y}{R}, \qquad e_{yy} = -\nu\frac{y}{R} = e_{zz}, \qquad e_{xy} = e_{yz} = e_{zx} = 0. \tag{12.5–3}$$

From these, we see that the equations of equilibrium, Eqs. (12.4–5), are satisfied. The equations of compatibility, Eqs. (6.3–4), are also satisfied. The boundary conditions on the lateral surface of the beam are $\overset{\nu}{T}_i = 0$. Since any normal to the lateral surface is perpendicular to the longitudinal axis x, the direction cosine ν_x vanishes; i.e., $\nu_x = 0$ on the lateral surface. Thus, the following boundary conditions are satisfied:

$$\overset{\nu}{T}_x = 0 = \sigma_{xx}\nu_x + \sigma_{xy}\nu_y + \sigma_{xz}\nu_z,$$
$$\overset{\nu}{T}_y = 0 = \sigma_{yx}\nu_x + \sigma_{yy}\nu_y + \sigma_{yz}\nu_z, \tag{12.5–4}$$
$$\overset{\nu}{T}_z = 0 = \sigma_{zx}\nu_x + \sigma_{yz}\nu_z + \sigma_{zz}\nu_z.$$

The boundary conditions at the ends of the beam are that the stress system must correspond to a pure bending moment, and without a resultant force. The stress system given by Eq. (12.5–1) does that, as discussed in Sec. 1.11. Hence, the solution is exact if the boundary stresses on the ends of the beam are distributed precisely in the manner specified by Eq. (12.5–1), because then all the differential equations and boundary conditions are satisfied.

One of the restrictions imposed in the derivation given in Sec. 1.11, namely, that the cross section of the beam has a plane of symmetry, can be removed. Let us consider, then, a prismatic beam with an arbitrary cross section, such as the one shown in Fig. 12.4. Assume the same stress and strain distribution as in Eqs. (12.5–1), (12.5–2), and (12.5–3). Suppose the boundary conditions, Eqs. (12.5–4), are also satisfied. The resultant axial force is zero again when the origin is taken at the centroid of the cross section. The resultant moment about the z-axis by the traction acting on the end section is given by the surface integral over the cross-section A:

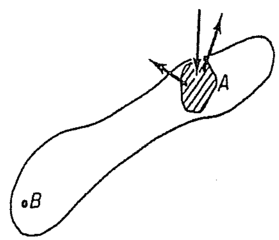

Figure 12.4 An unsymmetric cross section.

$$M_z = \int_A \sigma_{xx} y \, dA = \frac{E}{R}\int_A y^2 \, dA = \frac{EI_z}{R}.$$

It is the same as before, except that we have added the subscript z to show that the bending moment and the areal moment of inertia of the cross section are both taken about the z-axis. The resultant moment about the y-axis, however, is a new element to consider. It is given by an integration of the traction $\sigma_{xx} dA$ acting on an element of area dA situated at a distance z from the y-axis. Hence,

$$M_y = \int_A \sigma_{xx} z \, dA. \tag{12.5–5}$$

On substituting Eq. (12.5–1) into this equation, we obtain

$$M_y = \frac{E}{R}\int_A yz \, dA. \tag{12.5–6}$$

The integral is the negative of the *product of inertia of the cross-sectional area*:

$$P_{yz} = -\int_A yz \, dA \equiv -\iint_A yz \, dy \, dz. \tag{12.5–7}$$

Hence,

$$M_y = \frac{-EP_{yz}}{R}. \tag{12.5–8}$$

In case the beam cross section has a plane of symmetry in which the bending moment acts, we choose the xy-plane as that plane of symmetry; then $P_{yz} = 0$. It follows that $M_y = 0$, which shows that our solution in Sec. 1.11 is satisfactory. In the general case, we now choose the coordinate axes in such a way that the product of inertia vanishes. Then

$$P_{yz} = 0, \qquad M_y = 0, \tag{12.5–9}$$

and the moment vector is parallel to the z-axis with a magnitude equal to M_z.

The product of inertia vanishes if the y- and z-axes are the principal axes of inertia. Hence, in order that a moment acting in a plane produce bending in the same plane, it is necessary that the plane be a principal plane, i.e., one containing a principal axis of inertia of every cross section. Combining the requirements given by Eqs. (1.11–27) and (12.5–9), we see that *the coordinate axes y, z must be chosen as centroidal axes in the direction of the principal axes of the areal moments of inertia.*

Our verification is now complete. We have found that the stress system given by Eq. (12.5–1) is exact if y is measured from the neutral axis in the direction of a principal axis. The stress system satisfies the equations of equilibrium, the equations of compatibility, and the boundary conditions. Our intuitive assumption that plane sections remain plane is verified in this case.

More refined theories of bending can be found in many books, e.g., Sokolnikoff's *Mathematical Theory of Elasticity*.

12.6 BIOMECHANICS

Continuum mechanics can be applied to biology. Most biological materials can be considered as continua at suitable scales of observation. We have discussed the constitutive equations of a few biological tissues in Chap. 9. Most biological fluids and solids have nonlinear constitutive equations. The bone seems to be an exception, which functions in the small strain range and obeys Hooke's law. But the shape and internal structure of bones are very complex.

In important biological problems, fluid mechanics and solid mechanics are usually coupled together. For example, blood flows in blood vessels which have elastic walls. The heart pumps blood with a muscle. Hence the equations used in Chaps. 11 and 12 are usually coupled together in biomechanics.

Living tissues have a unique feature which is unmatched by nonliving materials. This is the feature of tissue remodeling under stress. By remodeling, the zero-stress state of the material changes, the constitutive equation changes, the mechanics changes. The following chapter is devoted to consider this new aspect of continuum mechanics.

PROBLEMS

12.2 An elementary theory of a circular cylindrical shaft subjected to torsion is given in Sec. 12.4. Let z be the axis of the shaft. Let the ends be $z = 0$ and $z = L$. A rectangular Cartesian frame of reference x, y, z is used. The displacement components in the x-, y-, and z-directions are u, v, and w, respectively. The elementary theory gives

$$u = -\alpha zy, \qquad v = \alpha zx$$

where α is the rate of twist angle per unit length of the shaft. The elementary theory does not say anything about the third component of displacement in the axial direction, w, which in general will not vanish if the shaft is non-circular. Let this unknown displacement be

$$w = \alpha\phi(x, y).$$

Using the equation of equilibriun (Navier's equation), find the equation satisfied by the function $\phi(x, y)$. This function is known as the *warping function.*

12.3 The following situation reminds a composite-material designer to pay attention to the question of the stability of a structure in operational condition. Consider a cantilever beam with a rectangular cross section. The beam is made of two strong rods embedded in a matrix. It is loaded by a force P parallel to the line joining the two rods in the cross section, as shown in Fig. P12.3. In practical application under a load, there is a probability that the beam will be twisted to failure. Twist will occur when the load P exceeds a critical value. Formulate a theory that will determine the critical value of the load P that will cause a torsional instability. According to your theory, how should such a composite beam be designed?

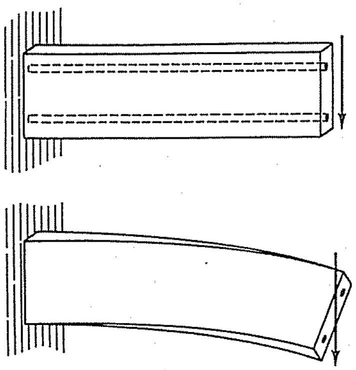

Figure P12.3 A beam of very narrow width may twist under load. The twist could be a critical problem for a beam reinforced with high-strength rods.

12.4 Consider a string of uniform density and material, stretched tightly between two posts (e.g., a violin string or a piano wire). The string is struck at a point. Vibration ensues. Formulate the problem mathematically. Give both the differential equations and the boundary conditions.

12.5 Consider a gong used in an orchestra. Formulate the mathematical problem of gong vibration.

12.6 Formulate a mathematical description of the clouds floating in the sky. How do they move about? Include enough parameters so that the great variety of things you see daily can be described and deduced.

12.7 An airplane flies in the air at a forward speed V relative to the ground. How does the wing maintain this flight? To answer this question, write down the field equations for the air and the wing, and the boundary conditions at the interface between the air and the wing. Present a full set of equations that would be sufficient to furnish a mathematical theory in principle.

12.8 The elastic waves in the rails as a train approaches are typical of waves in many dynamics problems. We can easily hear the impact of the wheels of the train (if we put our ears to the rail) long before the train can be seen. Then, as the train comes by, we can see the deflection of the rails under the wheels. Formulate the problem mathematically so that both of these features can be exhibited.

12.9 Feel the pulse on your wrist. It is a composite elastic wave in your artery. The most important component is undoubtedly the elastic response of the artery to the pressure wave in the blood. To a lesser extent, there must be other waves that are propagated along the arterial wall and caused by disturbances further upstream or downstream. Our arteries are elastic. Formulate a mathematical theory of pulse propagation. Leonhard Euler (1707–1783) formulated the problem and presented an analysis as early as 1775.

12.10 Galileo (1564–1642) proposed the following method for measuring the frequency of vibration of a gong. Attach a small, sharp, pointed knife to a slender rod. Pull the rod over the gong at a constant speed. The vibration of the gong will cause the rod to chatter. Examine the metal surface of the gong, and measure the spacing of the marks, from which the frequency may be calculated.

Explain whether this method will work. How would you compute the frequency? Formulate the problem mathematically from the point of view of the theory of elasticity. Assume a good musical gong to assure that the material is a linear elastic solid that obeys Hooke's law.

12.11 The phenomenon of chatter in machine tools is not unlike that in Galileo's gong experiment. Consider the problem of a high-speed lathe. Formulate the problem of chatter, which ruins a good machine's operation. Propose ways to alleviate the problem.

12.12 A beam vibrates. Write down the differential equation and boundary conditions for a vibrating beam and a method of determining the frequency of vibration of the beam.

12.13 A circular cylindrical shaft spins about its longitudinal axis at an angular speed ω radians/sec. The shaft is simply supported at both ends. Lateral vibrations are always possible when the shaft spins. However, if the rate of spin reaches a critical value, the lateral deformation becomes excessive, and so-called whirling sets in. Describe the phenomenon mathematically. Formulate the equations with which the critical whirling speed can be determined.

12.14 The shaft of an airplane propeller is subjected to both a tension and a torque. How would you propose to measure the stresses in the shaft in flight? How would you measure the power delivered to the propeller in flight?

FURTHER READING

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13 STRESS, STRAIN, AND ACTIVE REMODELING OF STRUCTURES

We use biological examples to bring out some fundamental issues of continuum mechanics: the zero-stress state, the changes in the zero-stress state and the constitutive equation due to remodeling of a material, the effect of stress and strain on remodeling, and the feedback dynamics of growth and resorption. Nonliving physical systems have these features also.

13.1 INTRODUCTION

In this last chapter, we discuss the mechanics of changes in materials. From the point of view of mechanics, there are three aspects of a solid body in which change plays a fundamental role: the zero-stress state, the constitutive equation, and the overall geometry of the body. Our discussion will focus on these aspects.

The mechanics of flow and deformation is called *rheology*. The literature on rheology is usually concerned only with flow and change in a given material or a given set of materials. The science in which growth and change in materials is a central concern is biology. In continuum mechanics rheology and biology are united. To illustrate the material-change aspects of continuum mechanics, examples can be picked from biology, because they are ubiquitous. In the discussion that follows, we often use the blood vessel as an example.

13.2 How to Discover the Zero-Stress State of Materials in a Solid Body

In the preceding chapters, it is assumed that when there is no external load acting on a body; the stress in the body is zero everywhere. We know, however, that this does not have to be the case; for example, we can sit, but tense up our muscles and create a lot of forces in our muscles and bones. Generally, the stress in a body when there is no external force is called *residual stress*. The effect of residual stress and strain can be dramatic, e.g., the relaxation of residual stress in the earth can cause an earthquake, and an unwanted thermal strain in a nuclear reactor might cause a meltdown.

The simplest way to discover residual stress in a solid body is to cut the body up. Cutting is introducing new surfaces on which the traction is zero. Cutting an unloaded body without residual stress will cause no strain. If strain changes by cutting, then there is residual stress.

Take a blood vessel as an example. If we cut an aorta twice by cross sections perpendicular to the longitudinal axis of the vessel, we obtain a ring. If we cut the ring radially, it will open up into a sector (see Fig. 13.1). By using equations of static equilibrium, we know that the stress resultants and stress moments are zero in the open sector. Whatever stress remains in the vessel wall must be locally in equilibrium. If we cut the open sector further and can show that no additional strain results, then we say that the sector is in the zero-stress state. For the rat artery Fung and Liu (1989) reported that this is the case.

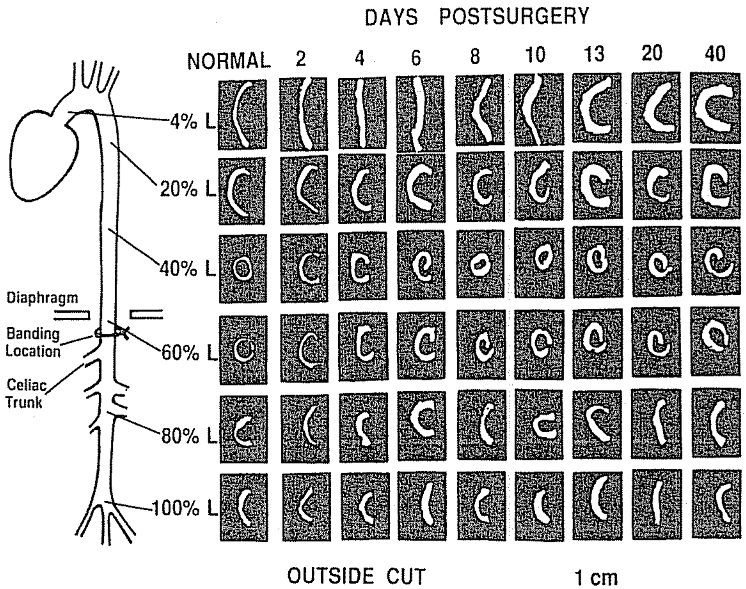

Figure 13.1 Photographs of the cross sections of a rat aorta at zero-stress state. The first column shows the zero-stress state of the aorta of normal rat. The rest shows the change of zero-stress state due to vessel remodeling after a sudden onset of hypertension. The photos are arranged from left to right according to days after surgery, and from top to bottom according to location on the aorta, expressed as distance from the heart in percentage of the total length of the aorta from the aortic valve to iliac bifurcation. The location of the metal clip used to induce hypertension is shown in the sketch at left. The arcs of the blood vessel wall do not appear smooth because of some tissue attached to the wall. In reading these photographs, one should mentally delete these tethered tissues. From Fung and Liu (1989).

Thus assured that the open sectors represent the zero-stress state of a blood vessel, we characterize each sector with an *opening angle*, which is defined as the angle subtended by two radii drawn from the mid-point of the inner wall to the tips of the inner wall of the open sector (see Fig. 13.2). A more complete picture of the zero-stress state of a normal young rat aorta is shown in the first column of Fig. 13.1 (Fung and Liu, 1989). The entire aorta was cut successively into many segments approximately one diameter long. Each segment was then cut radially. It was found that the opening angle varied along the rat aorta: It was about 160° in the ascending aorta, 90° in the arch, 60° in the thoracic region, 5° at the diaphragm level, and 80° toward the iliac bifurcation point.

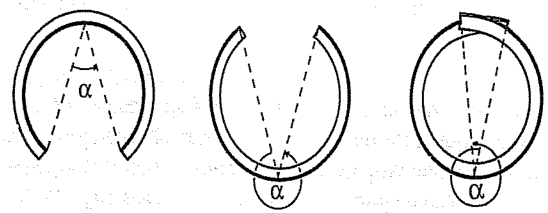

Figure 13.2 Definition of the opening angle. Sector represents a circumferential cross section of a blood vessel at zero-stress state. Angle subtended between two lines originating from the midpoint to the tips of the inner wall is the opening angle.

Following the common iliac artery down a leg of the rat, we found that the opening angle was around 100° in the iliac artery, dropped down in the popliteal artery region to 50°, and then rose again to about 100° in the tibial artery. In the medial plantar artery of the rat, the microarterial vessel of 50 mm diameter had an opening angle on the order of 100° (Liu and Fung, 1989).

There are similar, although not identical, spatial variations of opening angles in the aortas of the pig and dog (Han and Fung, 1991a). Also, there are significant opening angles in the pulmonary arteries (Fung and Liu, 1991), systemic and pulmonary veins (Xie et al., 1991), and trachea (Han and Fung, 1991b) of rats. Thus, we conclude that the zero-stress state of blood vessels and the trachea is shaped as sectors whose opening angles vary with their location on the vessel or trachea and with animal species. In other words, the zero-stress state in a body may vary from place to place. It then follows that the residual stress also varies spatially.

In industrial engineering, residual stresses are usually introduced into a solid body in the manufacturing process. Welding or riveting of metal parts under strain is a common cause of residual stress in airplanes, bridges, and machinery. Plastic deformation or creep in the metal-forming and machining process causes residual stress. Forced fitting is also a common cause. Straining steel rods is the way pre-stressed reinforced concrete beams are made. Heating an outer cylindrical shell to a higher temperature, fitting it to an inner shell, and then cooling the combination down to room temperature is the way gun barrels are made. The purpose is to induce a compressive residual stress in the inner wall and a tensile stress in the outer wall, so that when the gun is fired, the stress concentration at the inner wall of the barrel can be reduced. Shot peening to introduce a compressive residual stress on the outer surface of a metallic body is a way to increase the fatigue life of the body. Techniques using an ion or molecular beam to impregnate matter into the surface of a metallic or ceramic body can similarly introduce a compressive residual stress into a thin layer of the surface of the body to promote a longer service life. Most articles of industrial engineering have residual stress in them.

In living tissues, growth and change are natural. Every cellular or extracellular growth or resorption changes the zero-stress state of the tissue and introduces residual stress. In biological studies, it is easier to measure changes in the zero-stress state than to measure cellular activities in the tissue; hence, observed changes in residual strain are often used as a quantitative tool to study such activities.

Out of the great variety of examples in nature and industry, we shall use a few biological cases to illustrate the long-term effect of the stress in a body on the materials of the body. What in the short term is described by the stress-strain relationship becomes, in the long term, features associated with aging, remodeling, wear, tear, growth, and resorption. Biologists use the term *homeostasis* to describe the condition of normal life. They describe the state of a living organism at normal living conditions by a set of *homeostatic set points*. In a homeostatic condition, there is a certain range of stress in the body of the living organism. When the environment changes, the range of stress changes, the cells in the body respond by modifying themselves, and the tissue is remodeled. In effect, the zero-stress state of the body changes. In time, the mechanical properties of the tissue are also remodeled. We are familiar with these features in our own bodies. We know that homeostasis is not static, but a certain normal mode exists in a dynamic environment. The quantitative aspects will become clearer as we proceed in the sections that follow.

In machines and in nonliving physical objects, analogous features of homeostasis and remodeling may exist. These features are worthy of scientific study.

13.3 REMODELING THE ZERO-STRESS STATE OF A STRUCTURE: A BIOLOGICAL EXAMPLE OF ACTIVE REMODELING DUE TO CHANGE IN STRESS

In one study, hypertension was created in rats by constricting the abdominal aorta with a metal clip placed right above the celiac trunk. (See Fig. 13.3.) The clip severely constricted the aorta locally and reduced the normal cross-sectional area of the lumen by 97% (Fung and Liu, 1989; Liu and Fung, 1989). This caused a 20% step increase in blood pressure in the upper body and a 55% step decrease in blood pressure in the lower body immediately following surgery. Later, the blood pressure increased gradually, following the course shown in Fig. 13.4. In the upper body, the blood pressure rose rapidly at first and then more gradually, tending to an asymptote at about 75% above normal. In the lower body, the blood pressure rose to normal in about four days and then gradually increased further to an

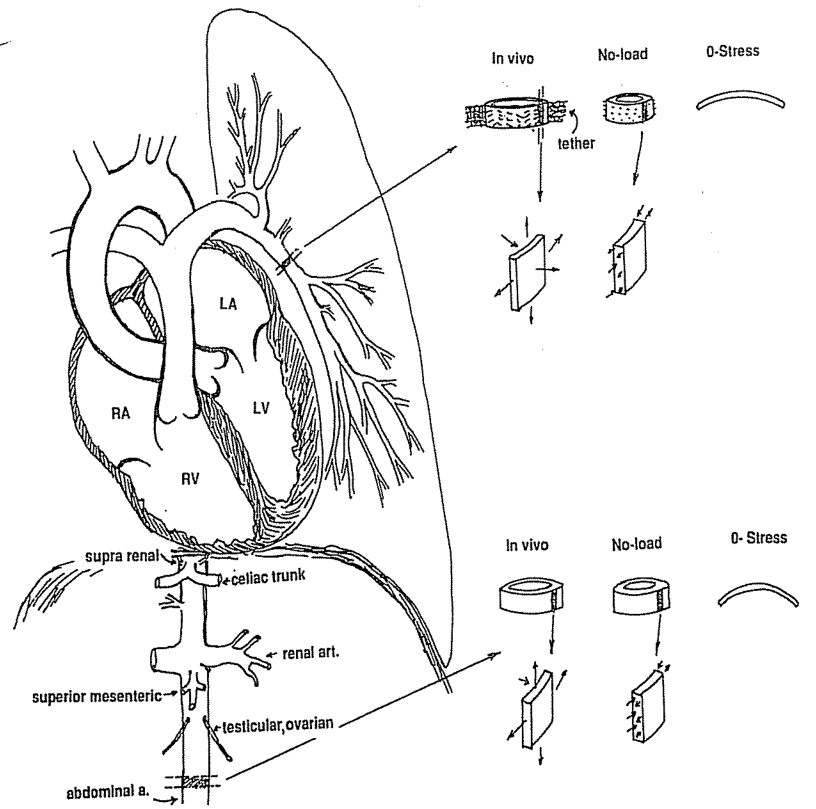

Figure 13.3 A sketch of the heart, aorta, and pulmonary arteries, the stresses in them, the zero-stress state, and the nomenclature of vessels mentioned in the text with regard to control of blood pressure by aortic constriction.

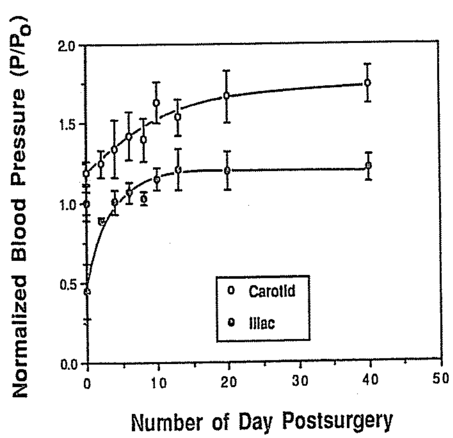

Figure 13.4 The course of change of blood pressure (normalized with respect to that before surgery) when a constriction is suddenly imposed on the aorta at a site below the diaphragm and above the celiac trunk shown in Fig. 13.3. From Fung and Liu (1989).

asymptotic value of 25% above normal. Parallel with these changes in blood pressure, the zero-stress state of the aorta changed as well. The changes are illustrated in Fig. 13.1, in which the location of any section on the aorta is indicated by the percentage distance of that section to the aortic valve measured along the aorta, divided by the total length of the aorta. Successive columns show the zero-stress configurations of the rat aorta at 0, 2, 4, . . . , 40 days after surgery. Successive rows refer to successive locations on the aorta.

The figure shows that following a sudden increase in blood pressure, the opening angles increased gradually, peaked in two to four days, and then decreased gradually to an asymptotic value. Variation with the location of the section on the aorta was great. The maximum change in the opening angle occurred in the ascending aorta, where the total swing of the opening angle was as large as 88°.

Thus, the blood vessel changed its opening angle in a few days following the change in blood pressure. Similar changes were found in the pulmonary arteries of rats after the onset of pulmonary hypertension by exposure to hypoxic gas containing 10% oxygen and 90% nitrogen at atmospheric pressure (Fung and Liu, 1991).

Thus, the zero-stress state of the blood vessel may be remodeled by an active biological process under the influence of changes in homeostatic stress.

13.4 CHANGE OF ZERO-STRESS STATE WITH TEMPERATURE: MATERIALS THAT "REMEMBER" THEIR SHAPES

The mechanical properties of a material may depend on many physical, chemical, and biological factors. We have illustrated a change in a material's zero-stress state due to a biological reaction to stress in the preceding section. Let us now consider a physical factor: temperature. It is well known that at any given state of stress, a change in temperature changes the strain, so that thermal stress may be regarded as caused by changing the zero-stress state through temperature variation.

There is, however, a more dramatic phenomenon in some materials. A hat made of a certain polymer can be folded for carrying and returned to good shape by heating. A medical device made of the same material has been used in Japan to close a patent ductus arteriosus in a young child. Ductus arteriosus is a vessel connecting a fetal heart to fetal lung, allowing blood to flow from the aorta to the pulmonary artery before birth. Normally it is closed immediately after birth. But sometimes it remains open and needs surgery. The device named above is shaped like a tiny umbrella, folded up, threaded to the duct by an endoarterial catheter, then opened up with a little squirt of fairly hot water from the catheter. The opened umbrella closes the ductus arteriosus.

Materials such as these, which appear to "remember" their shape, are materials whose zero-stress state changes with temperature. Alloys of copper-aluminum-nickel, copper-zinc-aluminum, iron-manganese-silicon, nickel-titanium, and polymers like polynorbornene have this property. For example, one nickel-titanium

(Ni-Ti) alloy, composed of equal number of atoms of nickel and titanium, and made into something at a higher temperature, can be deformed into some other shape at a lower temperature. If the deformed body is heated beyond a critical temperature, the Ni-Ti alloy will return to its original shape as manufactured and, if resisted, can generate a stress as high as 700 MPa (10^5 psi). This change is produced in the alloy by a change of crystalline phase known as a *martensitic transformation*. Martensite has a low yield stress, and can be deformed easily and reversibly by a crystalline process called *twining* of the atomic lattice. (See Fig. 13.5.) The martensitic transformation occurs over a range of temperature, above which the material is in the *austenitic phase*. When an austenite is cooled down, random twining occurs in the metal by random internal residual shear stress. Under an external

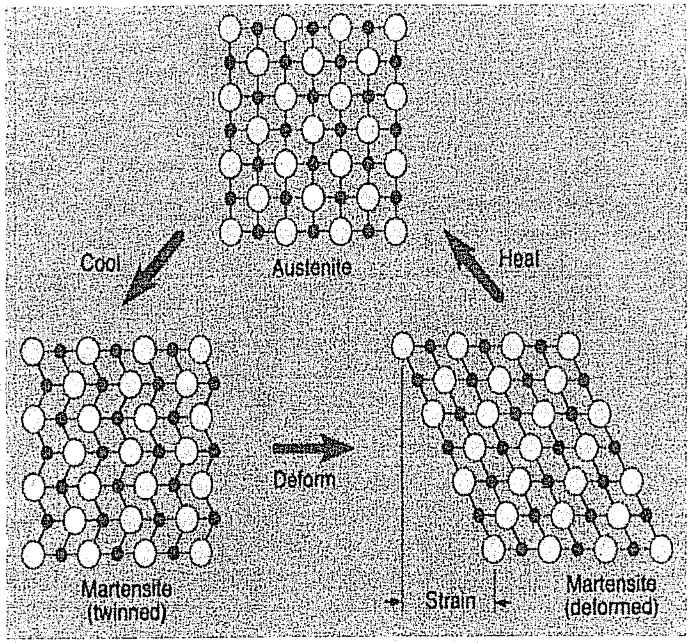

Figure 13.5 The mechanism of shape memory of a nickel-titanium alloy in the austenitic state at a higher temperature. The alloy is deformed at a lower temperature when the crystal structure is martensitic. The deformation in the martensite crystal is by *twining* which occurs under suitable shear stress and is reversible when the shear stress is reversed. If the temperature of the deformed martensite is raised to a level above a critical value, the crystalline structure of the alloy reverts to austenite and to the original shape of the body. From Tom Borden, "Shape-memory alloys: Forming a tight fit," *Mechanical Engineering*, Oct. 1991, p. 68. Reproduced by permission of the author and publisher.

shear load, a martensitic body can deform substantially and reversibly by twining. On heating the deformed martensite to a temperature at which the martensite crystal is transformed into austenite, the crystal reverts to its original shape, because austenite cannot accommodate the twining type of deformation.

The stress-strain curves for martensite and austenite are illustrated in Fig. 13.6. Deformations of martensite at strains greater than about 7% and austenite at strains greater than about 1% are plastic and irreversible. So for practical applications, one has to know the stress-strain curves, the ranges of elasticity and plasticity, the temperature at which austenite is first formed in martensite when heated, and the temperature at which martensite is first formed in austenite when cooled. With this knowledge, people have used Ni-Ti alloy for fastening machine parts, wiring teeth for orthodontic purposes, simulating the erection of an organ, and other phenomena.

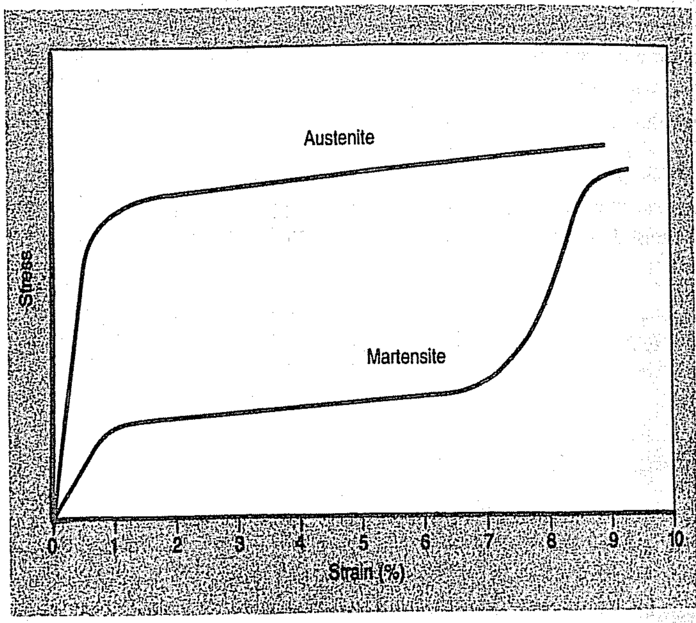

Figure 13.6 The stress-strain relationship of martensite and austenite crystals (tested at different temperature). From Tom Borden. *loc. cit.* By permission.

13.5 MORPHOLOGICAL AND STRUCTURAL REMODELING OF BLOOD VESSELS DUE TO A CHANGE IN BLOOD PRESSURE

The pressure of circulating blood varies from time to time and from place to place. What is normally referred to as the systemic blood pressure is the difference between the pressure of the blood in the aorta at the aortic valve and that in the

right atrium. This is the pressure difference that drives the entire "systemic" circulation throughout the body (the peripheral circulation system). The corresponding driving pressure for the pulmonary circulation is the difference between the pressure in the pulmonary artery at the pulmonic valve and the pressure in the left atrium. Both the systemic and the pulmonary circulation are characterized by systolic (in period of contraction of the heart) and diastolic (in period of dilatation of the heart) pressures. When these pressures change, the blood pressure in every vessel of the body changes. When the blood pressure changes, the stress in the blood vessel wall changes.

As is sketched in Fig. 13.3, in the in vivo condition at normal blood pressure, the circumferential stress is usually tensile and is the largest stress component in the vessel wall. The longitudinal stress components exist because the vessel is normally stretched in the axial direction. The radial stress component is compressive at the inner wall, where it is equal to the blood pressure, and gradually decreases to the pressure acting on the outer wall.

The systemic blood pressure can be changed in a number of ways: by drugs, by a high-salt diet, by constricting the flow of blood to the kidneys, etc. If the aorta is constricted severely by a stenosis above the renal arteries (Fig. 13.4), the aorta above the stenosis will become hypertensive. The aorta below the stenosis will become hypotensive at first, but the reduced blood flow to the kidneys will cause the kidneys to secrete more of the enzyme renin into the bloodstream and will raise the blood pressure. If the stenosis is below the kidney arteries and is sufficiently severe, then the lower body will become hypotensive. The pulmonary blood pressure can also be changed in a number of ways. A most convenient way is to change the oxygen concentration of the gas breathed by the animal. If the oxygen concentration is reduced from normal (i.e., if it becomes hypoxic), the smooth muscle cells in the pulmonary blood vessels contract, the vessel diameters are reduced, and the pulmonary blood pressure goes up. This is the reaction human beings who live at sea level encounter when they go to a high altitude.

The hypoxic hypertensive reaction occurs quite fast. If a rat is put into a low-oxygen chamber containing 10% oxygen and 90% nitrogen at atmospheric pressure at sea level, the systolic blood pressure in its lung will shoot up from the normal 2.0 kPa (15 mm Hg) to 2.9 kPa (22 mm Hg) within minutes, become further elevated to 3.6 kPa in a week, and then gradually rise to 4.0 kPa in a month. (The rat's systemic blood pressure remains essentially unchanged in the meantime.) Under such a rise in blood pressure in the lung, the pulmonary blood vessel remodels itself.

Figure 13.7 shows how fast this remodeling proceeds. In the figure, the photographs in each row refer to that segment of the pulmonary artery indicated by the leader line. The first photograph in the top row shows a cross section of the arterial wall of the normal three-month-old rat. The specimen was fixed at the no-load condition. In the figure, the endothelium is facing upward, with the vessel lumen on top. The endothelium is very thin—on the order of a few micrometers. The scale of 100 μm is shown at the bottom of the figure. The dark lines are elastin layers. The upper, darker half of the vessel wall is the media, the lower, lighter half of the vessel wall the adventitia. The second photograph in the first row shows

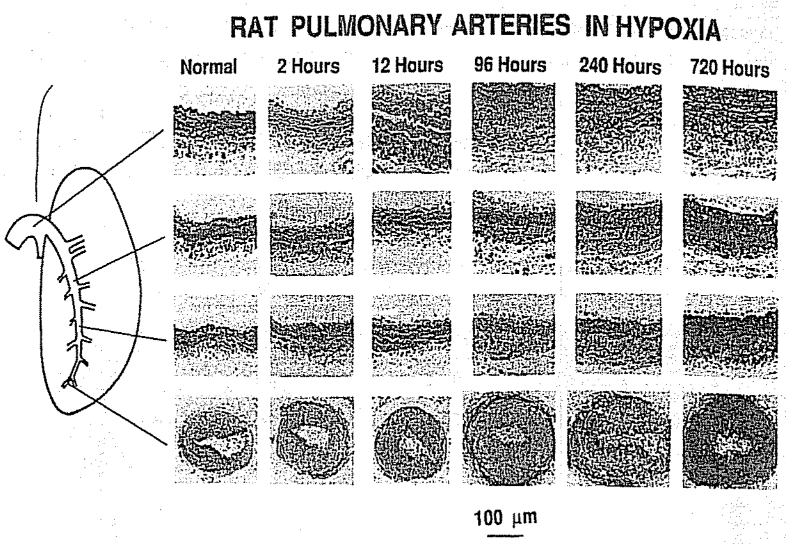

Figure 13.7 Photographs of histological slides from four regions of the main pulmonary artery of a normal rat and several hypertensive rats with different periods of hypoxia. Specimens were fixed at no-load condition. From Fung and Liu (1991).

a cross section of the main pulmonary artery two hours after exposure to lower oxygen pressure. There is evidence of small fluid vesicles and some accumulation of fluid in the endothelium and media. There is also a biochemical change of elastin staining on the vessel wall at this time. The third photograph shows the wall structure 10 hours later. Now the media is greatly thickened, while the adventitia has not changed very much. The fourth photograph shows that at 96 hours of exposure to hypoxia, the adventitia has thickened to about the same thickness as the media. The next two photos show the pulmonary arterial wall structure when the rat's lung is subjected to 10 and 30 days of lowered oxygen concentration. The major change in these later periods is the continued thickening of the adventitia.

The photographs in the second row show the progressive changes in the wall of a smaller pulmonary artery. The third and fourth rows are photographs of arteries of even smaller diameter. The inner diameter of the arteries in the fourth row is on the order of 100 μm, approaching the range of sizes of the arterioles. The remodeling of the vessel wall is evident in pulmonary arteries of all sizes. The maximum rate of change occurs in a day or two.

13.6 REMODELING OF MECHANICAL PROPERTIES

When the material in a blood vessel is changed during remodeling, its mechanical properties change. The mechanical properties of soft biological tissues can be described by the constitutive equations discussed in Secs. 9.5 and 9.7. Hence, we

expect that the constitutive equation, or at least its coefficients, will change with tissue remodeling. This is indeed the case, as we shall illustrate with an example.

For the blood vessel, the pseudoelasticity formulation of the constitutive equation, described in Secs. 9.4 and 9.5, applies. We assume that a *pseudoelastic strain energy function* exists, denoted by the symbol $\rho_0 W$ and expressed as a function of the nine components of strain E_{ij} ($i = 1, 2, 3, j = 1, 2, 3$), that is symmetric with respect to E_{ij} and E_{ji}, so that the stress components can be derived by a differentiation, namely,

$$S_{ij} = \frac{\partial \rho_0 W}{\partial E_{ij}} \tag{13.6-1}$$

Here, ρ_0 is the density of the material at the zero-stress state, W is the strain energy per unit mass, $\rho_0 W$ is the strain energy per unit volume, and E_{ij} are strains measured with respect to the material configuration in the zero-stress state.

With regard to the determination of $\rho_0 W$, two approaches may be taken. One is to regard the blood vessel wall as an incompressible material and derive $\rho_0 W$ as a function of E_{ij} in *three dimensions* (Chuong and Fung, 1983). The other is to assume that the blood vessel is a cylindrical body with axisymmetry in mechanical properties and limit oneself to axisymmetric loading and deformation. Then one would be concerned only with two strain components: the circumferential strain E_{11} and the longitudinal strain E_{22}. The radial strain is easily computed from the condition of incompressibility. This technique may be called a *two-dimensional approach*.

For the analytical representation of $\rho_0 W$ for arteries in the two-dimensional approach, a polynomial form has been used by Patel and Vaishnav (1972), a logarithmic form by Hayashi et al. (1971), and an exponential form by Fung et al. (1973, 1979, 1981), see references at the end of Chap. 9. According to Fung et al. (1979),

$$\rho_0 W = C \exp(a_1 E_{11}^2 + a_2 E_{22}^2 + 2a_4 E_{11} E_{22}) \tag{13.6-2}$$

where C, a_1, a_2, and a_4 are material constants, E_{11} is the circumferential strain, and E_{22} is the longitudinal strain, the last two referred to the zero-stress state.

Experiments have been done on rat arteries during the course of development of diabetes after a single injection of streptozocin. The results with the vessel wall treated as one homogeneous material are presented in Table 13.1, from Liu and

TABLE 13.1 COEFFICIENTS C, a_1, a_2, AND a_4 OF THE STRESS-STRAIN RELATIONSHIP OF THE THORACIC AORTA OF 20-DAY DIABETIC AND NORMAL RATS. a_4 WAS FIXED AS THE MEAN VALUE FROM THE NORMAL RATS.*

Group	C (n/cm²)	a_1	a_2	a_4
Normal Rats				
Mean ± SD	12.21 ± 3.32	1.04 ± 0.35	2.69 ± 0.95	0.0036
20-day Diabetic Rats				
Mean ± SD	15.32 ± 9.22	1.53 ± 0.92	3.44 ± 1.07	0.0036

*From Liu, S. Q., and Fung, Y. C. (1992).

Fung (1992). Clearly, the material constants change with the development of diabetes.

13.7 STRESS ANALYSIS WITH THE ZERO-STRESS STATE TAKEN INTO ACCOUNT

If the zero-stress state of a solid body is known, if the strain is infinitesimal, and if the constitutive equation is linear, then the principle of superposition applies, and the mathematical problem of the stress analysis of a body with residual stress is simply a sum of two linear problems: finding the residual stress without an external load and finding the stress under an external load but without residual strain. In this category fall the important classical theories of dislocation and thermal stress.

Nonlinearity introduced by a finite strain or constitutive equation makes the analysis of bodies with residual stress a distinctive subject. The nonlinear analysis is often very difficult. But if we know the zero-stress state and how it is related to the present state, then the analysis of stress in the body could be quite simple.

For example, consider an ileal artery whose cross section in vivo at a blood pressure of 16 kPa (120 mm Hg) is shown in Fig. 13.8. The cross sections under the no-load and zero-stress conditions are also shown in Fig. 13.8. From these figures, we can measure the length of the circumference of the inner wall of the vessel. Let the lengths at the zero-stress state, the no-load state, and the homeostatic (normal, in vivo) state be $L^{i\theta}_{0\text{-stress}}$, $L^{i\theta}_{\text{no-load}}$, and $L^{i\theta}_{\text{hom}}$, respectively, with the superscripts i indicating "inner" and θ indicating "circumferential," and the subscripts "0-stress," "no-load," and "hom" indicating the states of zero stress, no load, and homeostasis, respectively. Similarly, we can measure the circumferential length at

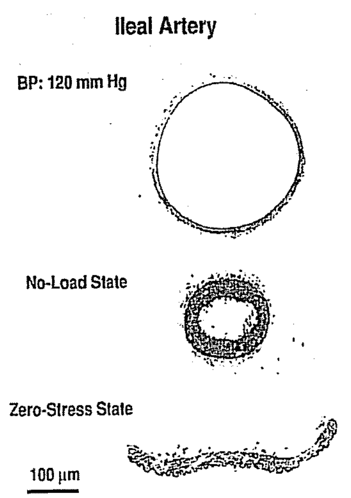

Figure 13.8 The shape of the cross section of an ileal artery of the rat at normal blood pressure (top), no load (middle), and zero stress (bottom).

the outer wall and obtain $L^{o\theta}_{0\text{-stress}}$, $L^{o\theta}_{\text{no-load}}$, and $L^{o\theta}_{\text{hom}}$, with the superscript o indicating "outer." From these, we obtain the stretch ratios

$$\lambda^{(i\theta)}_{\text{no-load}} = \frac{L^{(i\theta)}_{\text{no-load}}}{L^{(i\theta)}_{0\text{-stress}}}, \qquad \lambda^{(i\theta)}_{\text{hom}} = \frac{L^{(i\theta)}_{\text{hom}}}{L^{(i\theta)}_{0\text{-stress}}} \tag{13.7–1}$$

on the inner wall and

$$\lambda^{(o\theta)}_{\text{no-load}} = \frac{L^{(o\theta)}_{\text{no-load}}}{L^{(o\theta)}_{0\text{-stress}}}, \qquad \lambda^{(o\theta)}_{\text{hom}} = \frac{L^{(o\theta)}_{\text{hom}}}{L^{(o\theta)}_{0\text{-stress}}} \tag{13.7–2}$$

on the outer wall.

Typical raw data of the L's of an ileal artery, a medial plantar artery, and a pulmonary artery (branch 1) are given in Table 13.2. The computed stretch ratios are also listed in the table. These results may be compared with those obtained by a theoretical calculation of a hypothetical case in which the *no-load* and *homeostatic* configurations are identical with the real ones, but the residual strains are zero, so that the opening angle is zero and the zero-stress configuration is the same as the no-load configuration. In that case, the stretch ratios of the *no-load* case are unity, but those of the *homeostatic* vessel are

$$\lambda^{(i\theta)}_{\text{hom}} = \frac{L^{(i\theta)}_{\text{hom}}}{L^{(i\theta)}_{\text{no-load}}}, \qquad \lambda^{(o\theta)}_{\text{hom}} = \frac{L^{(o\theta)}_{\text{hom}}}{L^{(o\theta)}_{\text{no-load}}} \tag{13.7–3}$$

These are listed in the last two columns of Table 13.2.

The distribution of the circumferential residual stretch ratio in the vessel wall under the no-load condition is illustrated in the case of an ileal vessel (branch 1) in Fig. 13.9(a). It is seen that the residual stretch ratio is compressive in the inner wall and tensile in the outer wall. Under the conventional assumption that plane sections remain plane in bending, the stretch ratio distribution in the vessel wall is a straight line. In Fig. 13.9(b), the thick, nearly horizontal line shows the actual circumferential stretch ratio distribution in the blood vessel wall when the blood

TABLE 13.2 MEASURED CIRCUMFERENTIAL LENGTHS OF THE INNER AND OUTER WALLS OF RAT ILEAL ARTERY IN THE ZERO-STRESS STATE, IN THE NO-LOAD STATE, AND AT 80 AND 120 μm Hg; AND COMPARISON OF CIRCUMFERENTIAL STRETCH RATIOS COMPUTED ON TWO BASES: (A) RELATIVE TO THE ZERO-STRESS STATE, AND (B) RELATIVE TO THE NO-LOAD STATE.*

	Length, μm		Circumferential Stretch Ratio			
			Re zero-stress state		Re no-load state	
States	Inner Wall	Outer Wall	Inner Wall	Outer Wall	Inner Wall	Outer Wall
Zero Stress	743	963				
No load	590	1,091	0.79	1.13	1.0	1.0
80 mm Hg	1,017	1,281	1.37	1.33	1.72	1.17
120 mm Hg	1,023	1,286	1.38	1.34	1.73	1.18

*Data from Fung and Liu (1992).

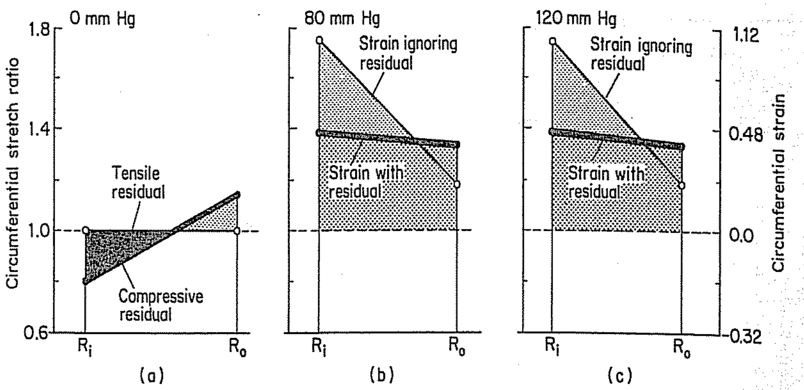

Figure 13.9 Circumferential stretch ratio distribution in an ileal artery (branch 1) whose dimensions are listed in Table 13.2. (a) Measured residual stretch ratio at no-load state. Residual strain can be read from nonlinear scale shown on right. R_i and R_o, inner and outer radius of vessel wall, respectively. Strain is compressive in inner wall region and tensile in outer wall region. (b) Thick, nearly horizontal line joining the solid dots shows measured circumferential stretch ratio (relative to the zero-stress state) at a blood pressure of 80 mm Hg; thinner inclined line joining open circles shows computed hypothetical circumferential stretch ratio when opening angle was ignored. (c) Corresponding strains at blood pressure 120 mm Hg. These curves show that huge errors result if residual strain is ignored. From Fung and Liu (1992). By permission.

pressure is 80 mm Hg, whereas the thinner, inclined line shows the hypothetical circumferential stretch ratio distribution at 80 mm Hg under the assumption that the opening angle is zero. The corresponding strains are all positive (tensile), but the great error caused by ignoring the residual strain (opening angle) is seen. The corresponding stretch ratio distributions in the vessel wall at a blood pressure of 120 mm Hg are illustrated in Fig. 13.9(c). It is clear from Fig. 13.9 that the errors caused by ignoring the residual strains are enormous. It is important to know the zero-stress state of a blood vessel.

The longitudinal stretch ratios from the no-load to the homeostatic condition measured on the specimens of Fig. 13.9 is about 1.35. No change in the ratio was detected experimentally upon cutting open a vessel segment under the no-load condition to the zero-stress state. Hence, the longitudinal stretch from the zero-stress state to the homeostatic state of the ileal vessel is also about 1.35. Finally, the radial stretch ratio can be computed from the condition of incompressibility of the vessel wall:

$$\lambda_r \lambda_\theta \lambda_z = 1. \qquad (13.7\text{–}4)$$

Thus, the strain state of the vessel is completely determined experimentally.

For arteries, the stresses increase as exponential functions of strains. Hence, if stresses were plotted in graphs corresponding to the strain distributions plotted in Fig. 13.9, a much greater error in stress would be seen as a consequence of ignoring the opening angle.

13.8 STRESS-GROWTH RELATIONSHIP

Biological tissue growth can be affected by many things: nutrition, growth factors (enzymes), the physical and chemical environment, and diseases, as well as stress and strain. If other things were equal, then a stress-growth law will emerge.

A stress-growth relationship has clinical applications in the understanding of diseases, healing, and rehabilitation. If a stress-growth law is known for certain organs, then surgeons can use it to plan surgery on those organs, engineers can use it for tissue engineering, manufacturers of prostheses will have guidance, and physical therapists, athletes, and educators will know the relation between exercise and body development.

Tissue engineering is a field dedicated to making artificial substitutes for living tissues. It is a technology based on molecular biology, cell biology, and organ physiology. To master tissue engineering, one must know how the health of tissues is maintained, improved, or failed in relation to stress and strain.

Machines, in general, do not have the ability to remodel themselves, but such an ability is clearly desirable in some circumstances. It is not beyond the engineer's imagination to conceive of machines with the ability to remodel themselves, but the direction is a totally new one for engineers to think about.

Readers interested in this subject may find the references listed at the end of this chapter helpful. A fairly comprehensive introduction to the mechanics of tissue remodeling is given in Fung (1990), which contains an extensive list of references. In the medical field, bone remodeling has been studied for a long time. Meyer's paper was dated 1867. Wolff's law was proposed in 1869. Papers by Carter and Wong (1988), Cowin (1986), and Fukada (1977) may serve as entry to the current literature. In the preceding sections we used blood vessels to illustrate the features of tissue remodeling: changes in the zero-stress state, structure and arterial composition, constitutive equations, and stress and strain distributions. We could have used bone for this purpose; but changes in soft tissues are more visible and take place faster than those in bone. The getting together of the time constants for tissue remodeling, stress relaxation, strain creep, fluid movement, and mass transport serves to bring biology and mechanics closer together. The papers by Chuong and Fung (1986), Fung (1991), Hayashi and Takamizawa (1989), Takamizawa and Hayashi (1987), Vaishnav and Vossoughi (1987), and Omens and Fung (1990) are relevant to soft tissue mechanics. The book edited by Skalak and Fox (1988) is a collection of papers presented at a tissue engineering conference. There is a large amount of literature on the biology and medicine of tissue remodeling. The papers by Cowan and Crystal (1975), and Meyrick and Reid (1980) are excellent examples.

PROBLEM

13.1 Membranes within Living Cells. Within a cell, membranes are ubiquitous, but their mechanical properties are virtually unknown. As a theoretical concept, intracellular membranes may be assumed to have surface tension, stretching elasticity, shear elasticity, and bending rigidity. The tension and shear are associated with membrane area and deformation, the bending rigidity is associated with the change in curvature of the membrane.

A surface in three-dimensional Euclidean space has two principal curvatures at every point. The sum of the two principal curvatures is called the *mean curvature* of the surface; the product of the two principal curvatures is called the *Gaussian curvature*. One may assume that the energy state of the membrane depends on the mean and Gaussian curvatures. Now, propose a strain energy function for an intracellular membrane. Then solve a mathematical problem: find a *minimal* surface of finite area but zero mean curvature everywhere.

An answer given by Reinhard Lipowsky, in *Nature*, vol. 349, p. 478, Feb. 1991 is shown in Fig. P13.1. Do you think Lipowsky's surface is minimal? What kind of energy state would the surface have? If one wants to claim that a minimal surface has a minimum energy level, how should energy of the membrane be related to the surface area, and the mean and Gaussian curvatures?

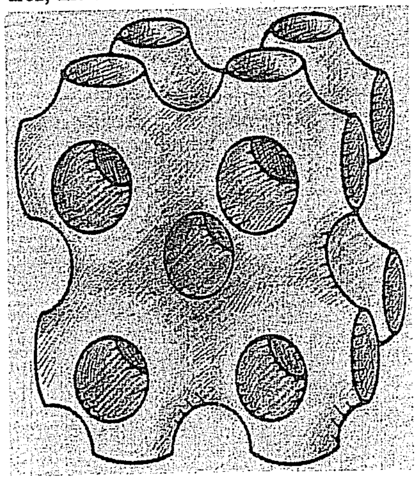

Figure P13.1 Lipowsky's surface.

What kind of surface has zero Gaussian curvature everywhere? Is a developable surface one of zero Gaussian curvature? Are all surfaces with zero Gaussian curvature developable?

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