10

VARIATIONAL CALCULUS,

ENERGY THEOREMS,

SAINT-VENANT'S PRINCIPLE

There are at least three important reasons for taking up the calculus of variations in the study of continuum mechanics.

1. Because basic minimum principles exist, which are among the most beautiful of theoretical physics.
2. The field equations (ordinary or partial differential equations) and the associated boundary conditions of many problems can be derived from variational principles. In formulating an approximate theory, the shortest and clearest derivation is usually obtained through variational calculus.
3. The "direct" method of solution of variational problems is one of the most powerful tools for obtaining numerical results in practical problems of engineering importance.

In this chapter we shall discuss several variational principles and their applications.

A brief introduction to the calculus of variations is furnished below. Those readers who are familiar with the mathematical techniques of the calculus of variations may skip over the first six sections.

10.1. MINIMIZATION OF FUNCTIONALS

The calculus of variations is concerned with the minimization of functionals. If $u(x)$ is a function of x, defined for x in the interval (a, b), and if I is a quantity defined by the integral

$$I = \int_a^b [u(x)]^2 \, dx,$$

then the value of I depends on the function $u(x)$ as a whole. We may indicate this dependence by writing $I[u(x)]$. Such a quantity I is said to be a *functional* of $u(x)$. Physical examples of functionals are the total kinetic energy of a flow field and the strain energy of an elastic body.

The basic problem of the calculus of variations may be illustrated by the following example. Let us consider a functional $J[u]$ defined by the integral

$$J[u] = \int_a^b F(x, u, u') \, dx. \tag{1}$$

We shall give our attention to *all* functions $u(x)$ which are continuous and differentiable, with continuous derivatives $u'(x)$ and $u''(x)$ in the interval $a < x < b$, and satisfying the boundary conditions

$$u(a) = u_0, \qquad u(b) = u_1, \tag{2}$$

where u_0 and u_1 are given numbers. We assume that the function $F(x, u, u')$ in Eq. (1) is continuous and differentiable with respect to x, and all such u, and u', up to all second-order partial derivatives, which are themselves continuous.

Among all functions $u(x)$ satisfying these continuity conditions and boundary values, we try to find a special one $u(x) = y(x)$, with the property that $J[u]$ attains a minimum when $u(x) = y(x)$, with respect to a sufficiently small neighborhood of $y(x)$. The neighborhood (h) of $y(x)$ is defined as follows. If h is a positive quantity, a function $u(x)$ is said to lie in the neighborhood (h) of $y(x)$ if the inequality

$$|y(x) - u(x)| < h \tag{3}$$

holds for all x in (a, b). The situation is illustrated in Fig. 10.1:1.

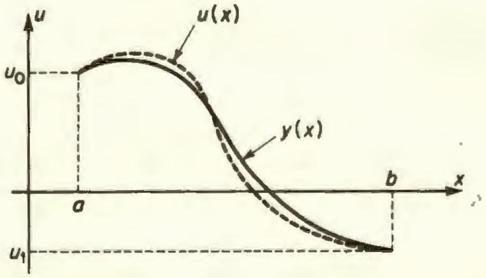


Fig. 10.1:1. Functions $u(x)$ and $y(x)$.

Let us assume that the problem posed above has a solution, which will be designated by $y(x)$; i.e., there exists a function $y(x)$ such that the inequality

$$J[y] < J[u] \tag{4}$$

holds for all functions $u(x)$ in a sufficiently small neighborhood (h) of $y(x)$. Let us exploit the necessary consequences of this assumption.

Let $\eta(x)$ be an arbitrary function with the properties that $\eta(x)$ and its derivatives $\eta'(x)$, $\eta''(x)$ are continuous in the interval $a < x < b$, and that

$$\eta(a) = \eta(b) = 0. \tag{5}$$

Then the function

(6) $$u(x) = y(x) + \epsilon\eta(x)$$

satisfies all the continuity conditions and boundary values specified at the beginning of this section. In fact, any function $u(x)$ satisfying these conditions can be represented with some function $\eta(x)$ in this manner. For a sufficiently small $\delta > 0$, this function $u(x)$ belongs, for all ϵ with $|\epsilon| < \delta$, to a prescribed neighborhood (h) of $y(x)$. Now we introduce the function

(7) $$\phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F[x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)]\,dx.$$

Since $y(x)$ is assumed to be known, $\phi(\epsilon)$ is a function of ϵ for any specific $\eta(x)$. According to (4), the inequality

(8) $$\phi(0) < \phi(\epsilon)$$

must hold for all ϵ with $|\epsilon| < \delta$. In other words, $\phi(\epsilon)$ attains a minimum at $\epsilon = 0$. The function $\phi(\epsilon)$ is differentiable with respect to ϵ. Therefore, the necessary condition for $\phi(\epsilon)$ to attain a minimum at $\epsilon = 0$ must follow,

(9) $$\phi'(0) = 0,$$

where a prime indicates a differentiation with respect to ϵ. Now, a differentiation under the sign of integration yields

(10)
$$\phi'(\epsilon) = \int_a^b [F_u(x, y + \epsilon\eta, y' + \epsilon\eta')\eta(x) + F_{u'}(x, y + \epsilon\eta, y' + \epsilon\eta')\eta'(x)]\,dx.$$

where F_u, $F_{u'}$ indicates $\partial F/\partial u$, $\partial F/\partial u'$, respectively. Integrating the last integral by parts, we obtain

(11)
$$\phi'(\epsilon) = \int_a^b \left[F_u(x, y + \epsilon\eta, y' + \epsilon\eta') - \frac{d}{dx}F_{u'}(x, y + \epsilon\eta, y' + \epsilon\eta')\right]\eta(x)\,dx + F_{u'}(x, y + \epsilon\eta, y' + \epsilon\eta')\eta(x)\Big|_a^b$$

The last term vanishes according to (5). $\phi'(\epsilon)$ must vanish at $\epsilon = 0$, at which u equals y. Hence, we obtain the equation

(12) $$0 = \phi'(0) = \int_a^b \left[F_y(x, y, y') - \frac{d}{dx}F_{y'}(x, y, y')\right]\eta(x)\,dx,$$

which must be valid for an arbitrary function $\eta(x)$.

Equation (12) leads immediately to Euler's differential equation, by virtue of the following "fundamental lemma of the calculus of variations:"

LEMMA. *Let $\psi(x)$ be a continuous function in $a < x < b$. If the relation*

(13) $$\int_a^b \psi(x)\eta(x)\,dx = 0$$

holds for all functions $\eta(x)$ which vanish at $x = a$ and b and are continuous together with their first 2n derivatives, where n is a positive integer, then $\psi(x) \equiv 0$.

Proof. This lemma is easily proved indirectly. We shall show first that $\psi(x) \equiv 0$ in the open interval $a < x < b$. Let us suppose that this statement is not true, that $\psi(x)$ is different from zero, say positive, at $x = \xi$, where ξ lies in the open interval. Then, according to the continuity of $\psi(x)$, there must exist an interval $\xi - \delta < x < \xi + \delta$ (with $a < \xi - \delta$, $\xi + \delta < b$, $\delta > 0$), in which $\psi(x)$ is positive. Now we take the function

(14) $$\eta(x) = \begin{cases} (x - \xi + \delta)^{4n}(x - \xi - \delta)^{4n} \text{ in } \xi - \delta < x < \xi + \delta, \\ 0 \text{ elsewhere,} \end{cases}$$

where n is a positive integer. (See Fig. 10.1:2.) This function $\eta(x)$ satisfies the continuity and boundary conditions specified above. But the choice of $\eta(x)$ as defined by (14) will make $\int_a^b \psi(x)\eta(x)\,dx > 0$, in contradiction to the hypothesis. Thus, the hypothesis is untenable, and $\psi(x) \equiv 0$ in $a < x < b$. According to the continuity of $\psi(x)$, we get $\psi(x) = 0$ also in $a < x < b$. Q.E.D. We remark that the lemma can be extended to hold equally well for multiple integrals.

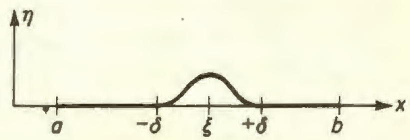


Fig. 10.1:2. The function $\eta(x)$.

From Eq. (12), it follows immediately from the lemma that the factor in front of $\eta(x)$ must vanish:

(15a) ▲ $$F_y(x, y, y') - \frac{d}{dx}F_{y'}(x, y, y') = 0, \qquad a < x < b.$$

This is a differential equation that $y(x)$ must satisfy and is known as Euler's equation. Written out *in extenso*, we have

(15b) ▲ $$\frac{d^2y}{dx^2}\frac{\partial^2 F}{\partial y'\,\partial y'}(x, y, y') + \frac{dy}{dx}\frac{\partial^2 F}{\partial y'\,\partial y}(x, y, y') + \frac{\partial^2 F}{\partial y'\,\partial x}(x, y, y') - \frac{\partial F}{\partial y}(x, y, y') = 0.$$

Should the problem be changed to finding the necessary condition for $J[u]$ to attain a maximum with respect to a sufficiently small neighborhood (h) of $y(x)$, the same result would be obtained. Hence, the result: *The validity of Euler's differential Eq.* (15) *is a necessary condition for a function $y(x)$ to furnish an extremum of the functional $J[u]$ with respect to a sufficiently small neighborhood (h) of $y(x)$.*

The satisfaction of Euler's equation is a necessary condition for $J[u]$ to attain an extremum; but it is not a sufficient condition. The question of sufficiency is rather involved; an interested reader must refer to treatises on calculus of variations such as those listed in Bib. 10.1, on p. 494.†

Now a point of notation. It is customary to call $\epsilon\eta(x)$ the *variation* of $u(x)$ and write

(16) $$\epsilon\eta(x) = \delta u(x).$$

It is also customary to define the first variation of the functional $J[u]$ as

(17) $$\delta J = \epsilon\, \phi'(\epsilon).$$

On multiplying both sides of Eq. (11) by ϵ, it is evident that it can be written as

(18) $$\delta J = \int_a^b \left[F_u(x, u, u') - \frac{d}{dx} F_{u'}(x, u, u') \right] \delta u(x)\, dx + F_{u'}(x, u, u')\, \delta u(x) \Big|_a^b .$$

This is analogous to the notation of the differential calculus, in which the expression $df = \epsilon f'(x)$ for an arbitrary small parameter ϵ is called the "differential of the function $f(x)$." It is obvious that δJ depends on the function $u(x)$ and its variation $\delta u(x)$. Thus, a necessary condition for $J[u]$ to attain an extremum when $u(x) = y(x)$ is the vanishing of the first variation δJ for all variations δu with $\delta u(a) = \delta u(b) = 0$.

Example 1.

$$I[u] = \int_a^b (1 + u'^2)\, dx = \min, \qquad u(a) = 0, \qquad u(b) = 1.$$

The Euler equation is

$$-\frac{d}{dx} F_{y'} = \frac{d}{dx} 2y' = 2y'' = 0.$$

Hence $y(x)$ is a straight line passing through the points $(a, 0)$ and $(b, 1)$.

Example 2.

Which curve minimizes the following functional?

$$I[u] = \int_0^{\pi/2} [(u')^2 - u^2]\, dx, \qquad u(0) = 0, \qquad u(\tfrac{1}{2}\pi) = 1.$$

Ans. $y = \sin x$.

† Any consideration of the sufficient conditions requires the concept of the second variations and the examination of its positive or negative definiteness. Several conditions are known; they are similar to, but more complex than, the corresponding conditions for maxima or minima of ordinary functions. It is useful to remember that many subtleties exist in the calculus of variations. A physicist or an engineer rarely worries about the mathematical details. His minimum principles are established on physical grounds and the existence of a solution is usually taken for granted. Mathematically, however, many examples can be constructed to show that a functional may not have a maximum or a minimum, or that a solution of the Euler equation may not minimize the functional. An engineer should be aware of these possibilities.

Example 3. Minimum surface of revolution

Find $y(x)$ that minimizes the functional

$$I[u] = 2\pi \int_a^b u\sqrt{1 + u'^2}\, dx, \qquad u(a) = A, \qquad u(b) = B.$$

Ans. The Euler equation can be integrated to give

$$y\sqrt{1 + y'^2} - \frac{yy'^2}{\sqrt{1 + y'^2}} = c_1.$$

Let $y' = \sinh t$, then $y = c_1 \cosh t$ and

$$dx = \frac{dy}{y'} = \frac{c_1 \sinh t\, dt}{\sinh t} = c_1\, dt, \qquad x = c_1 t + c_2.$$

Hence, the minimum surface is obtained by revolving a curve with the following parametric equations about the x-axis,

$$x = c_1 t + c_2, \qquad y = c_1 \cosh t, \quad \text{or} \quad y = c_1 \cosh \frac{x - c_2}{c_1},$$

which is a family of catenaries. The constants c_1, c_2 can be determined from the end points.

10.2. FUNCTIONAL INVOLVING HIGHER DERIVATIVES OF THE DEPENDENT VARIABLE

In an analogous manner one can treat the variational problem connected with the functional

(1) $$J[u] = \int_a^b F(x, u, u^{(1)}, \ldots, u^{(n)})\, dx$$

involving $u(x)$ and its successive derivatives $u^{(1)}(x), u^{(2)}(x), \ldots, u^{(n)}(x)$, for $a < x < b$. To state the problem concisely we denote by D the set of all real functions $u(x)$ with the following properties:

(2a) $u(x), \ldots, u^{(2n)}(x)$ continuous in $a < x < b$,

(2b) $u(a) = \alpha_0, \quad u^{(\nu)}(a) = \alpha_\nu, \quad \nu = 1, \ldots, n - 1,$

(2c) $u(b) = \beta_0, \quad u^{(\nu)}(b) = \beta_\nu, \quad \nu = 1, \ldots, n - 1,$

where α_0, β_0, α_ν and β_ν are given numbers. A function $u(x)$ which possesses these continuity properties and boundary values is said to be in the set D. We assume that the function $F(x, u, u^{(1)}, \ldots, u^{(n)})$ has continuous partial derivatives up to the order $n + 1$ with respect to the $n + 2$ arguments $x, u, u^{(1)}, \ldots, u^{(n)}$ for u in the set D. We seek a function $u(x) = y(x)$ in D so that $J[u]$ is a minimum (or a maximum) for $u(x) = y(x)$, with respect to a sufficiently small neighborhood (h) of $y(x)$.

Let $\eta(x)$ be an arbitrary function with the properties

$$(3)\qquad \begin{aligned}&\eta(x), \eta^{(1)}(x), \ldots, \eta^{(2n)}(x) \text{ continuous in } a < x < b,\\ &\eta(a) = \ldots = \eta^{(n-1)}(a) = \eta(b) = \ldots = \eta^{(n-1)}(b) = 0.\end{aligned}$$

Then the function $u(x) = y(x) + \epsilon\eta(x)$ belongs to the set D for all ϵ. For sufficiently small ϵ's this function belongs to a prescribed neighborhood (h) of $y(x)$. Once again, we introduce the function

$$(4)\qquad \phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F(x, y + \epsilon\eta, y^{(1)} + \epsilon\eta^{(1)}, \ldots, y^{(n)} + \epsilon\eta^{(n)})\, dx.$$

The assumption that $y(x)$ minimizes $J[u]$ leads to the necessary condition $\phi'(0) = 0$. So we obtain by differentiating under the sign of integration, integrating by parts, and using Eqs. (3), the result

$$(5)\qquad \begin{aligned}0 = \phi'(0) &= \int_a^b \left\{\sum_{\nu=0}^{n}\left[\frac{\partial}{\partial y^{(\nu)}} F(x, y, y^{(1)}, \ldots, y^{(n)})\right]\eta^{(\nu)}(x)\right\} dx\\ &= \int_a^b \left\{\sum_{\nu=0}^{n}(-1)^{\nu}\frac{d^{\nu}}{dx^{\nu}}\left[\frac{\partial}{\partial y^{(\nu)}} F(x, y, y^{(1)}, \ldots, y^{(n)})\right]\right\}\eta(x)\, dx,\end{aligned}$$

which holds for an arbitrary function $\eta(x)$ satisfying (3). According to the lemma proved in Sec. 10.1, we obtain the Euler equation

$$(6)\ \blacktriangle\qquad \sum_{\nu=0}^{n}(-1)^{\nu}\frac{d^{\nu}}{dx^{\nu}}\left[\frac{\partial}{\partial y^{(\nu)}} F(x, y, y^{(1)}, \ldots, y^{(n)})\right] = 0,$$

which is a necessary condition for the minimizing (or maximizing) function $y(x)$.

The notations δu, δJ, etc., introduced in Sec. 10.1, can be extended to this case by obvious changes.

Example

Find the extremal of $J[u] = \int_0^1 (1 + u''^2)\, dx$ satisfying the boundary conditions

$$u(0) = 0, \qquad u'(0) = 1, \qquad u(1) = 1, \qquad u'(1) = 1.$$

Ans. $y = x$.

10.3. SEVERAL UNKNOWN FUNCTIONS

The method used in the previous sections can be extended to more complicated functionals. For example, let

$$(1)\qquad J[u_1, u_2, \ldots, u_m] = \int_a^b F(x, u_1, \ldots, u_m; u_1', \ldots, u_m')\, dx$$

be a functional depending on m functions $u_1(x), \ldots, u_m(x)$. We assume that the functions $F(x, u_1, \ldots, u_m')$, $u_1(x), \ldots, u_m(x)$ are twice differentiable, and that the boundary values of $u_1, \ldots, u_m$ are given at $x = a$ and b. We seek a special set of functions $u_\mu(x) = y_\mu(x)$, $\mu = 1, \ldots, m$, in order that $J[u_1, \ldots, u_m]$ attains a minimum (or a maximum) when $u_\mu(x) = y_\mu(x)$, with respect to a sufficiently small neighborhood of the $y_\mu(x)$; i.e., for all $u_\mu(x)$ satisfying the relation

$$|y_\mu(x) - u_\mu(x)| < h_\mu, \qquad h_\mu > 0, \qquad \mu = 1, 2, \ldots, m.$$

Again it is easy to obtain the necessary conditions. Let the set of functions $y_1(x), \ldots, y_m(x)$ be a solution of the variational problem. Let $\eta_1(x), \ldots, \eta_m(x)$ be an arbitrary set of functions with the properties

$$(2)\qquad \begin{aligned}&\eta_\mu(x), \eta_\mu'(x), \eta_\mu''(x) \text{ continuous in } a < x < b,\\ &\eta_\mu(a) = \eta_\mu(b) = 0, \qquad \mu = 1, \ldots, m.\end{aligned}$$

Then consider the function

$$(3)\qquad \begin{aligned}\phi(\epsilon_1, \ldots, \epsilon_m) &= J[y_1 + \epsilon_1\eta_1, \ldots, y_m + \epsilon_m\eta_m]\\ &= \int_a^b F(x, y_1 + \epsilon_1\eta_1, \ldots, y_m + \epsilon_m\eta_m; y_1' + \epsilon_1\eta_1', \ldots, y_m' + \epsilon_m\eta_m')\, dx.\end{aligned}$$

Since $y_1(x), \ldots, y_m(x)$ minimize (or maximize) $J[y_1 + \epsilon\eta_1, \ldots, y_m + \epsilon_m\eta_m]$, the following inequality must hold for sufficiently small $\epsilon_1, \ldots, \epsilon_m$:

$$(4)\qquad \phi(\epsilon_1, \ldots, \epsilon_m) \geq \phi(0, \ldots, 0), \qquad [\text{or } \phi(\epsilon_1, \ldots, \epsilon_m) \leq \phi(0, \ldots, 0)].$$

The corresponding necessary conditions are

$$(5)\qquad \left(\frac{\partial\phi}{\partial\epsilon_\mu}\right) = 0, \quad \text{for} \quad \epsilon_1 = \epsilon_2 = \ldots = \epsilon_m = 0, \quad \mu = 1, \ldots, m,$$

which lead to

$$(6)\qquad \begin{aligned}0 = \int_a^b [&F_{y_\mu}(x, y_1, \ldots, y_m, y_1', \ldots, y_m')\eta_\mu(x)\\ &+ F_{y'_\mu}(x, y_1, \ldots, y_m, y_1', \ldots, y_m')\eta_\mu'(x)]\, dx,\\ &\qquad \mu = 1, \ldots, m.\end{aligned}$$

Using integration by parts and the conditions (2), we obtain

$$(7)\qquad \begin{aligned}0 = \int_a^b [&F_{y_\mu}(x, y_1, \ldots, y_m, y_1', \ldots, y_m')\\ &- \frac{d}{dx} F_{y'_\mu}(x, y_1, \ldots, y_m, y_1', \ldots, y_m')]\eta_\mu(x)\, dx\\ &+ F_{y'_\mu}(x, y_1, \ldots, y_m, y_1', \ldots, y_m')\eta_\mu(x)\Big|_a^b,\\ &\qquad \mu = 1, \ldots, m.\end{aligned}$$

Equation (7) must hold for any set of functions $\eta(x)$ satisfying (2). By the lemma of Sec. 10.1 we have the following Euler equations which must be satisfied by $y_1(x), \dots, y_m(x)$ minimizing (or maximizing) the functional (1):

$$(8) \quad \blacktriangle \quad F_{y_\mu}(x, y_1, \dots, y_m; y_1', \dots, y_m') - \frac{d}{dx} F_{y'_\mu}(x, y_1, \dots, y_m; y_1', \dots, y_m') = 0, \qquad \mu = 1, \dots, m.$$

As a generalization of the variational notations given in Sec. 10.1 the expression

$$(12) \qquad \delta J = \sum_{\mu=1}^{m} \epsilon_\mu \left(\frac{\partial \phi}{\partial \epsilon_\mu}\right)$$

is called the "first variation of the functional."

Variational problems for functionals involving higher derivatives of $u_1, \dots, u_m$ can be treated in the same way.

Example

Find the extremals of the functional

$$J[y, z] = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\, dx,$$

$$y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1, \quad z(0) = 0, \quad z\left(\frac{\pi}{2}\right) = -1.$$

The Euler equations are

$$y'' - z = 0, \qquad z'' - y = 0.$$

Eliminating z, we have $y^{\text{iv}} - y = 0$. Hence,

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x,$$
$$z = y'' = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x.$$

From the boundary conditions we obtain the solution

$$y = \sin x, \qquad z = -\sin x.$$

10.4. SEVERAL INDEPENDENT VARIABLES

Consider the functional

$$(1) \qquad J[u] = \iint_G F(x, y, u, u_x, u_y)\, dx\, dy,$$

where G is a finite, closed domain of the x, y-plane with a boundary curve C which has a piecewise continuously turning tangent. The function $F(x, y, u, u_x, u_y)$ is assumed to be twice continuously differentiable with respect to its

five arguments. Let D be the set of all functions $u(x, y)$ with the following properties:

$$(2) \qquad D: \begin{cases} \text{(a)} & u(x, y), \; u_x(x, y), \; u_y(x, y), \; u_{xx}(x, y), \\ & u_{xy}(x, y), \; u_{yy}(x, y) \text{ continuous in } G, \\ \text{(b)} & u(x, y) \text{ prescribed on } C. \end{cases}$$

We now seek a special function $u(x, y) = v(x, y)$ in the set D, which minimizes (or maximizes) the functional $J[u]$.

Let $v(x, y)$ be a solution of this variational problem. Let $\eta(x, y)$ denote an arbitrary function with the properties

$$(3) \qquad \begin{cases} \text{(a)} & \eta(x, y) \text{ has continuous derivatives up to the second order in } G, \\ \text{(b)} & \eta(x, y) = 0 \text{ for } (x, y) \text{ on the boundary } C. \end{cases}$$

A consideration of the function

$$(4) \qquad \phi(\epsilon) = J[v + \epsilon\eta],$$

which attains an extremum when $\epsilon = 0$, again leads to the necessary condition

$$(5) \qquad \frac{d\phi}{d\epsilon}(0) = 0,$$

i.e., explicitly,

$$(6) \qquad 0 = \iint_G \{F_v(x, y, v, v_x, v_y)\eta + F_{v_x}(x, y, v, v_x, v_y)\eta_x + F_{v_y}(x, y, v, v_x, v_y)\eta_y\}\, dx\, dy.$$

The last two terms can be simplified by Gauss' theorem after rewriting (6) as†

$$0 = \iint_G \left\{F_v \cdot \eta + \frac{\hat\partial}{\partial x}(F_{v_x} \cdot \eta) - \eta \frac{\hat\partial}{\partial x} F_{v_x} + \frac{\hat\partial}{\partial y}(F_{v_y} \cdot \eta) - \eta \frac{\hat\partial}{\partial y} F_{v_y}\right\} dx\, dy.$$

An application of Gauss' theorem to the sum of the second and fourth terms in the integrand gives

$$(7) \qquad 0 = \iint_G \left\{F_v - \frac{\hat\partial}{\partial x} F_{v_x} - \frac{\hat\partial}{\partial y} F_{v_y}\right\}\eta(x, y)\, dx\, dy + \int_C \{F_{v_x} \cdot n_1(s) + F_{v_y} \cdot n_2(s)\}\eta\, ds$$

Here $n_1(s)$ and $n_2(s)$ are the components of the unit outer-normal vector $\mathbf{n}(s)$ of C.

† The symbol $(\hat\partial/\partial x)F_{v_x}$ means that F_{v_x} should be considered as a function of x and y, e.g.,

$$\frac{\hat\partial}{\partial x} F_{v_x} = \frac{\partial}{\partial x} F_{v_x} + \frac{\partial F_{v_x}}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F_{v_x}}{\partial v_x}\frac{\partial v_x}{\partial x} + \frac{\partial F_{v_x}}{\partial v_y}\frac{\partial v_y}{\partial x}.$$

The line integral in (7) vanishes according to (3). The function $\eta(x, y)$ in the surface integral is arbitrary. The generalized lemma of Sec. 10.1 then leads to the Euler equations

$$F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y} = 0;$$

i.e.,

$$(9) \quad \blacktriangle \quad \frac{\partial F}{\partial v} - \frac{\partial^2 F}{\partial v_x \, \partial x} - \frac{\partial^2 F}{\partial v_y \, \partial y} - \frac{\partial^2 F}{\partial v_x \, \partial v} \frac{\partial v}{\partial x} - \frac{\partial^2 F}{\partial v_y \, \partial v} \frac{\partial v}{\partial y} - \frac{\partial^2 F}{\partial v_x^2} \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 F}{\partial v_x \, \partial v_y} \frac{\partial^2 v}{\partial x \, \partial y} - \frac{\partial^2 F}{\partial v_y^2} \frac{\partial^2 v}{\partial y^2} = 0.$$

Equation (9) is a necessary condition for a function $v(x, y)$ in the set D minimizing (or maximizing) the functional (1).

In the same way one may treat variational problems connected with functionals which involve more than two independent variables, higher derivatives, and several unknown functions.

Example 1

$$J[u] = \iint_D (u_x^2 + u_y^2) \, dx \, dy,$$

with a boundary condition that u is equal to an assigned function $f(x, y)$ on the boundary C of the domain D. The Euler equation is the Laplace equation $v_{xx} + v_{yy} = 0$.

Example 2

$$J[u] = \iint_D (u_{xx}^2 + u_{yy}^2 + 2u_{xy})^2 \, dx \, dy = \min.$$

Then v must satisfy the biharmonic equation

$$\frac{\partial^4 v}{\partial x^4} + 2 \frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4} = 0.$$

10.5. SUBSIDIARY CONDITIONS—LAGRANGIAN MULTIPLIERS

In many problems, we are interested in the extremum of a function or functional under certain subsidiary conditions. As an elementary example, let us consider a function $f(x, y)$ defined for all (x, y) in a certain domain G. Suppose that we are interested in the extremum of $f(x, y)$, not for all points in G, but only for those points in G which satisfy the relation

$$(1) \qquad \phi(x, y) = 0.$$

Thus, if the domain G and the curve $\phi(x, y) = 0$ are as shown in Fig. 10.5:1, then we are interested in finding the extremum of $f(x, y)$ among all points

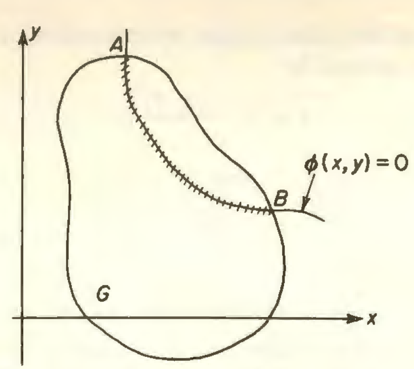


Fig. 10.5:1. Illustrating a subsidiary condition.

that lie on the segment of the curve AB. For conciseness of expression, we shall call such a subdomain—the segment AB—G_ϕ.

Let us find the necessary conditions for $f(x, y)$ to attain an extreme value at a point $(\bar{x}, \bar{y})$ in G_ϕ, with respect to all points of a sufficiently small neighborhood of $(\bar{x}, \bar{y})$ belonging to G_ϕ.

Let us assume that the function $\phi(x, y)$ has continuous partial derivatives with respect to x and y, and that at the point $(\bar{x}, \bar{y})$ not both derivatives are zero; say,

$$(2) \qquad \phi_y(\bar{x}, \bar{y}) \neq 0.$$

Then, according to a fundamental theorem on implicit functions, there exists a neighborhood of $\bar{x}$, say $\bar{x} - \delta < x < \bar{x} + \delta$ $(\delta > 0)$, where the equation $\phi(x, y) = 0$ can be solved uniquely in the form

$$(3) \qquad y = g(x).$$

The function $g(x)$ so defined is single-valued and differentiable, and $\phi[x, g(x)] = 0$ is an identity in x. Hence

$$(4) \qquad 0 = \phi_x[x, g(x)] + \phi_y[x, g(x)] \frac{dg(x)}{dx}.$$

Now, let us consider the problem of the extremum value of $f(x, y)$ in G_ϕ. According to (3), in a sufficiently small neighborhood of $(\bar{x}, \bar{y})$, y is an implicit function of x, and $f(x, y)$ becomes

$$(5) \qquad \mathscr{F}(x) = f[x, g(x)].$$

If $\mathscr{F}(x)$ attains an extreme value at $\bar{x}$, then the first derivative of $\mathscr{F}(x)$ vanishes at $\bar{x}$; i.e.,

$$(6) \qquad 0 = \frac{d\mathscr{F}}{dx}(\bar{x}) = f_x(\bar{x}, \bar{y}) + f_y(\bar{x}, \bar{y}) \frac{dg}{dx}(\bar{x}).$$

But the derivative dg/dx can be eliminated between (6) and (4). The result, together with (1), constitutes the necessary condition for an extremum of $f(x, y)$ in G_ϕ.

The formalism will be more elegant by introducing a number λ, called *Lagrange's multiplier*, defined by

$$\lambda = -\frac{f_y(\bar{x}, \bar{y})}{\phi_y(\bar{x}, \bar{y})}. \tag{7}$$

A combination of (4), (6), and (7) gives

$$f_x(\bar{x}, \bar{y}) + \lambda\phi_x(\bar{x}, \bar{y}) = 0, \tag{8}$$

while (7) may be written as

$$f_y(\bar{x}, \bar{y}) + \lambda\phi_y(\bar{x}, \bar{y}) = 0. \tag{9}$$

Equations (1), (8), and (9) are necessary conditions for the function $f(x, y)$ to attain an extreme at a point $(\bar{x}, \bar{y})$, where $\phi_x^2 + \phi_y^2 > 0$. These conditions constitute three equations for the three "unknowns" x, y, and λ.

These results can be summarized in the following manner. We introduce a new function

$$F(x, y; \lambda) = f(x, y) + \lambda\phi(x, y). \tag{10}$$

If the function $f(x, y)$ has an extreme value at the point $(\bar{x}, \bar{y})$ with respect to G_ϕ, and if

$$[\phi_x(\bar{x}, \bar{y})]^2 + [\phi_y(\bar{x}, \bar{y})]^2 > 0, \tag{11}$$

then there exists a certain number $\bar{\lambda}$ so that the three partial derivatives of $F(x, y; \lambda)$ with respect to x, y, and λ are zero at $(\bar{x}, \bar{y}, \bar{\lambda})$:

$$\begin{aligned} \frac{\partial F}{\partial x}(\bar{x}, \bar{y}, \bar{\lambda}) &= f_x(\bar{x}, \bar{y}) + \bar{\lambda}\phi_x(\bar{x}, \bar{y}) = 0, \\ \frac{\partial F}{\partial y}(\bar{x}, \bar{y}, \bar{\lambda}) &= f_y(\bar{x}, \bar{y}) + \bar{\lambda}\phi_y(\bar{x}, \bar{y}) = 0, \\ \frac{\partial F}{\partial \lambda}(\bar{x}, \bar{y}, \bar{\lambda}) &= \phi(\bar{x}, \bar{y}) = 0. \end{aligned} \tag{12}$$

If there exist points (x', y') in G with $\phi(x', y') = 0$ and $\phi_x(x', y') = \phi_y(x', y') = 0$, additional considerations are necessary.

In the formulation (10) and (12), the theorem can be generalized to the case of n variables $x_1, \ldots, x_n$ and several subsidiary conditions $0 = \phi_1(x_1, \ldots, x_n) = \ldots = \phi_m(x_1, \ldots, x_n)$, $m < n$.

The application of the Lagrange multiplier method to the minimization of functionals follows a similar reasoning.

As an example, let us consider the classical problem of geodesics: to find the line of minimal length lying on a given surface $\phi(x, y, z) = 0$ and joining

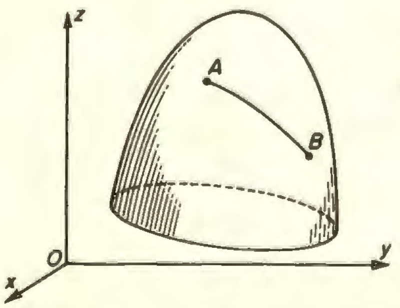


Fig. 10.5:2. A problem of geodesics.

two given points on this surface (Fig. 10.5:2). Here we must minimize the functional

$$I = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx, \tag{13}$$

with $y(x)$, $z(x)$ satisfying the condition $\phi(x, y, z) = 0$. This problem was solved in 1697 by Johann Bernoulli, but a general method of solution was given by L. Euler and J. Lagrange.

We consider a new functional

$$I^*[y, z, \lambda] = \int_{x_0}^{x_1} [\sqrt{1 + y'^2 + z'^2} + \lambda(x)\phi(x. y, z)]\, dx \tag{14}$$

and use the method of Sec. 10.3 to determine the functions $y(x)$, $z(x)$, and $\lambda(x)$ that minimizes I^*. The necessary conditions are

$$\lambda\frac{\partial\phi}{\partial y} - \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = 0, \tag{15}$$

$$\lambda\frac{\partial\phi}{\partial z} - \frac{d}{dx}\frac{z'}{\sqrt{1 + y'^2 + z'^2}} = 0, \tag{16}$$

$$\phi(x, y, z) = 0. \tag{17}$$

This system of equations determines the functions $y(x)$, $z(x)$, and $\lambda(x)$.

10.6. NATURAL BOUNDARY CONDITIONS

In previous sections we considered variational problems in which the admissible functions have prescribed values on the boundary. We shall now consider problems in which no boundary values are prescribed for the admissible functions. Such problems lead to the *natural boundary conditions*.

Consider again the functional (10.1:1) but now omit the boundary condition (10.1:2). Following the arguments of Sec. 10.1, we obtain the necessary condition (10.1:11),

$$0 = \int_a^b \left\{F_y - \frac{d}{dx}F_{y'}\right\}\eta(x)\, dx + F_{y'} \cdot \eta(x)\Big|_a^b. \tag{1}$$

This equation must hold for all arbitrary functions $\eta(x)$ and, in particular, for arbitrary functions $\eta(x)$ with $\eta(a) = \eta(b) = 0$. This leads at once to the Euler equation (10.1:15)

$$F_y(x, y, y') - \frac{d}{dx}F_{y'}(x, y, y') = 0, \qquad \text{in } a < x < b. \tag{2}$$

In contrast to Sec. 10.1, however, the last term $F_{y'} \cdot \eta(x)\,|_a^b$, does not vanish by prescription. Hence, by Eqs. (1) and (2), we must have

$$(F_{y'} \cdot \eta)_{x=b} - (F_{y'} \cdot \eta)_{x=a} = 0 \tag{3}$$

for all functions $\eta(x)$. But now $\eta(a)$ and $\eta(b)$ are arbitrary. Taking two functions $\eta_1(x)$ and $\eta_2(x)$ with

$$\eta_1(a) = 1, \qquad \eta_1(b) = 0; \qquad \eta_2(a) = 0, \qquad \eta_2(b) = 1; \tag{4}$$

we get, from (3),

$$F_{y'}[a, y(a), y'(a)] = 0, \qquad F_{y'}[b, y(b), y'(b)] = 0. \tag{5}$$

The conditions (5) are called the ***natural boundary conditions*** of our problem. They are the boundary conditions which must be satisfied by the function $y(x)$ if the functional $J[u]$ reaches an extremum at $u(x) = y(x)$, provided that $y(a)$ and $y(b)$ are entirely arbitrary.

Thus, if the first variation of a functional $J[u]$ vanishes at $u(x) = y(x)$, and if the boundary values of $u(x)$ at $x = a$ and $x = b$ are arbitrary, then $y(x)$ must satisfy not only the Euler equation but also the natural boundary conditions which, in general, involve the derivatives of $y(x)$. In contrast to the ***natural*** boundary conditions, the conditions $u(a) = \alpha$, $u(b) = \beta$, which specify the boundary values of $u(x)$ at $x = a$ and b, are called ***rigid boundary conditions***.

The concept of natural boundary conditions is also important in a more general type of variational problem in which boundary values occur explicitly in the functionals. It can be generalized also to functionals involving several dependent and independent variables and higher derivatives of the dependent variables.

The idea of deriving natural boundary conditions for a physical problem is of great importance and will be illustrated again and again below. See, for example, Sec. 10.8, 11.2.

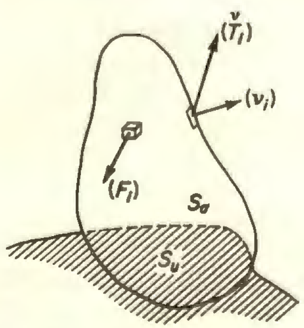


Fig. 10.7:1. Notations.

10.7. THEOREM OF MINIMUM POTENTIAL ENERGY UNDER SMALL VARIATIONS OF DISPLACEMENTS

Let a body be *in static equilibrium* under the action of specified body and surface forces. (Fig. 10.7:1). The boundary surface S shall be assumed to consist of two parts, S_σ and S_u, with the following boundary conditions.

Over S_σ: The surface traction $\overset{\nu}{T}_i$ is prescribed.
Over S_u: The displacement u_i is prescribed.

We assume that there exists a system of displacements u_1, u_2, u_3 satisfying the Navier's equations of equilibrium and the given boundary conditions. Let us consider a class of arbitrary displacements $u_i + \delta u_i$ consistent with the constraints imposed on the body. Thus δu_i must vanish over S_u, but it is arbitrary over S_σ. We further restrict δu_i to be triply differentiable and to be

of such an order of magnitude that the material remains elastic. Such arbitrary displacements δu_i are called *virtual displacements*.

Let us assume that static equilibrium prevails and compute the "virtual work" done by the body force F_i per unit volume and the surface force $\overset{\nu}{T}{}_i^*$ per unit area:

$$\int_V F_i\,\delta u_i\,dv + \int_S \overset{\nu}{T}{}_i^*\,\delta u_i\,dS.$$

On substituting $\overset{\nu}{T}{}_i^* = \sigma_{ij}\nu_j$ and transforming according to Gauss' theorem, we have

$$\begin{aligned}\int_S \overset{\nu}{T}{}_i^*\,\delta u_i\,dS &= \int_S \sigma_{ij}\,\delta u_i\nu_j\,dS \\ &= \int_V (\sigma_{ij}\,\delta u_i)_{,j}\,dv \\ &= \int_V \sigma_{ij,j}\,\delta u_i\,dv + \int_V \sigma_{ij}\delta u_{i,j}\,dv.\end{aligned} \tag{1}$$

According to the equation of equilibrium the first integral on the right-hand side is equal to $-\int F_i\,\delta u_i\,dv$. On account of the symmetry of σ_{ij}, the second integral may be written as

$$\int_V \sigma_{ij}[\tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i})]\,dv = \int_V \sigma_{ij}\,\delta e_{ij}\,dv.$$

Therefore, (1) becomes

$$\int_V F_i\,\delta u_i\,dv + \int_{S_\sigma} \overset{\nu}{T}{}_i^*\,\delta u_i\,dS = \int_V \sigma_{ij}\,\delta e_{ij}\,dv. \tag{2 ▲}$$

This equation expresses the *principle of virtual work*. The surface integral needs only be integrated over S_σ, since δu_i vanishes over the surface S_u where boundary displacements are given.

If the *strain-energy function* $W(e_{11}, e_{12}, \ldots)$ exists, so that $\sigma_{ij} = \partial W/\partial e_{ij}$, (Sec. 6.1), then it can be introduced into the right-hand side of Eq. (2). Since

$$\int_V \sigma_{ij}\,\delta e_{ij}\,dv = \int_V \frac{\partial W}{\partial e_{ij}}\,\delta e_{ij}\,dv = \delta\int_V W\,dv, \tag{3}$$

the principle of virtual work can be stated as

$$\delta\int_V W\,dv - \int_V F_i\,\delta u_i\,dv - \int_{S_\sigma} \overset{\nu}{T}{}_i^*\,\delta u_i\,dS = 0. \tag{4 ▲}$$

Further simplification is possible if the body force F_i and the surface tractions $\overset{\nu}{T}{}_i^*$ are *conservative* so that

$$F_i = -\frac{\partial G}{\partial u_i}, \qquad \overset{\nu}{T}{}_i^* = -\frac{\partial g}{\partial u_i}. \tag{5}$$

The functions $G(u_1, u_2, u_3)$ and $g(u_1, u_2, u_3)$ are called the potential of F_i and $\overset{\nu}{T}{}_i^*$, respectively. In this case,

$$(6) \qquad -\int_V F_i\, \delta u_i\, dv - \int_{S_\sigma} \overset{\nu}{T}{}_i^*\, \delta u_i\, dS = \delta\int_V G\, dv + \delta\int_S g\, dS.$$

Then Eq. (4) may be written as

$$(7) \blacktriangle \qquad \delta\mathscr{V} = 0,$$

where

$$(8) \blacktriangle \qquad \mathscr{V} \equiv \int_V (W + G)\, dv + \int_S g\, dS$$

The function $\mathscr{V}$ is called the *potential energy of the system.* This equation states that the potential energy has a stationary value in a class of admissible variations δu_i of the displacements u_i in the equilibrium state. Formulated in another way, it states that, *of all displacements satisfying the given boundary conditions, those which satisfy the equations of equilibrium are distinguished by a stationary (extreme) value of the potential energy.* For a rigid-body, W vanishes and the familiar form is recognized. We emphasize that the linearity of the stress-strain relationship has not been invoked in the above derivation, so that *this principle is valid for nonlinear, as well as linear, stress-strain law, as long as the body remains elastic.*

That this stationary value is a miminum in the neighborhood of the natural, unstrained state follows the assumption that the strain energy function is positive definite in such a neighborhood (see Secs. 12.4, 12.5). This can be shown by comparing the potential energy $\mathscr{V}$ of the actual displacements u_i with the energy $\mathscr{V}'$ of another system of displacements $u_i + \delta u_i$ satisfying the condition $\delta u_i = 0$ over S_u. We have

$$(9) \qquad \mathscr{V}' - \mathscr{V} = \int [W(e_{11} + \delta e_{11}, \ldots,) - W(e_{11}, \ldots,)]\, dv - \int F_i\, \delta u_i\, dv - \int \overset{\nu}{T}{}_i^*\, \delta u_i\, dS.$$

Expanding $W(e_{11} + \delta e_{11}, \ldots,)$ into a power series, we have

$$(10) \qquad W(e_{11} + \delta e_{11}, \ldots,) = W(e_{11}, \ldots,) + \frac{\partial W}{\partial e_{ij}}\, \delta e_{ij} + \frac{1}{2}\frac{\partial^2 W}{\partial e_{ij}\, \partial e_{kl}}\, \delta e_{ij}\, \delta e_{kl} + \cdots.$$

A substitution into (9) yields, up to the second order in δe_{ij},

$$(11) \qquad \mathscr{V}' - \mathscr{V} = \int_V \frac{\partial W}{\partial e_{ij}}\, \delta e_{ij}\, dv - \int_V F_i\, \delta u_i\, dv - \int_S \overset{\nu}{T}{}_i^*\, \delta u_i\, dS + \int_V \frac{1}{2}\frac{\partial^2 W}{\partial e_{ij}\, \partial e_{kl}}\, \delta e_{ij}\, \delta e_{kl}\, dv.$$

The sum of the terms in the first line on the right-hand side vanishes on account of (4). The sum in the second line is positive for sufficiently small values of strain δe_{ij} and can be seen as follows. Let us set $e_{ij} = 0$ in Eq. (10). The constant term is immaterial. The linear term must vanish because $\partial W/\partial e_{ij} = \sigma_{ij}$, which must vanish as $e_{ij} \to 0$. Hence, up to the second order,

$$(12) \qquad \frac{1}{2}\frac{\partial^2 W}{\partial e_{ij}\, \partial e_{kl}}\, \delta e_{ij}\, \delta e_{kl} = W(\delta e_{ij}).$$

Therefore, (11) becomes

$$(13) \qquad \mathscr{V}' - \mathscr{V} = \int W(\delta e_{ij})\, dv.$$

If $W(\delta e_{ij})$ is positive definite, then the last line in (11) is positive, and

$$(14) \qquad \mathscr{V}' - \mathscr{V} > 0$$

and that $\mathscr{V}$ is a minimum is proved. Accordingly, our principle is called the *principle of minimum potential energy.* The equality sign holds only if all δe_{ij} vanish, i.e., if the virtual displacements consist of a virtual rigid-body motion. If there were three or more points of the body fixed in space, such a rigid-body motion would be excluded, and $\mathscr{V}$ is a strong minimum; otherwise it is a weak minimum.

To recapitulate, we remark again that the variational principle (2) is generally valid; (4) is established whenever the strain energy function $W(e_{11}, e_{12}, \ldots,)$ exists; and (7) is established when the potential energy $\mathscr{V}$ can be meaningfully defined, but the fact that $\mathscr{V}$ is a minimum for "actual" displacements is established only in the neighborhood of the stable natural state, where W is positive definite.

Conversely, we may show that the variational principle gives the equations of elasticity. In fact, starting from Eq. (7) and varying u_i, we have

$$(15) \qquad \delta\mathscr{V} = \int_V \frac{\partial W}{\partial e_{ij}}\, \delta e_{ij}\, dv - \int_V F_i\, \delta u_i\, dv - \int_S \overset{\nu}{T}{}_i^*\, \delta u_i\, dS = 0.$$

But

$$\int \frac{\partial W}{\partial e_{ij}}\, \delta e_{ij}\, dv = \frac{1}{2}\int \sigma_{ij}(\delta u_{i,j} + \delta u_{j,i})\, dv = -\int \sigma_{ij,j}\, \delta u_i\, dv + \int \sigma_{ij}\nu_j\, \delta u_i\, dS.$$

Hence,

$$(16) \qquad \int_V (\sigma_{ij,j} + F_i)\, \delta u_i\, dv + \int_S (\sigma_{ij}\nu_j - \overset{\nu}{T}{}_i^*)\, \delta u_i\, dS = 0,$$

which can be satisfied for arbitrary δu_i if

$$(17) \qquad \sigma_{ij,j} + F_i = 0 \qquad \text{in } V$$

and either

$$(18) \qquad \delta u_i = 0 \qquad \text{on } S_u \text{ (rigid boundary condition)},$$

or

$$\overset{\nu}{T}{}_i^* = \sigma_{ij}\nu_j \qquad \text{on } S_\sigma \text{ (natural boundary condition).} \tag{19}$$

The one-to-one correspondence between the differential equations of equilibrium and the variational equation is thus demonstrated; for we have first derived (7) from the equation of equilibrium and then have shown that, conversely, Eqs. (17)–(19) necessarily follow (7).

The commonly encountered external force systems in elasticity are conservative systems in which the body force F_i and surface tractions $\overset{\nu}{T}_i$ are independent of the elastic deformation of the body. In this case, $\mathscr{V}$ is more commonly written as

$$\mathscr{V} \equiv \int_V W\,dv - \int_V F_i u_i\,dv - \int_{S_\sigma} \overset{\nu}{T}{}_i^* u_i\,dS. \tag{20}$$

A branch of mechanics in which the external forces are in general non-conservative is the theory of aeroelasticity. In aeroelasticity one is concerned with the interaction of aerodynamic forces and elastic deformation. The aerodynamic forces depend on the flow and the deformation of the entire body, not just the local deformation; thus in general it cannot be derived from a potential function. The principle of virtual work, in the form of Eq. (4), is still applicable to aeroelasticity.

Problem 10.1. The minimum potential energy principle states that elastic equilibrium is equivalent to the condition $J = \min$, where

(a) $$J = \int_V [W(e_{ij}) + F_i u_i]\,dV + \int_{S_\sigma} \overset{\nu}{T}_i u_i\,dS, \qquad \text{(varying } u_i\text{);}$$

(b) $$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}).$$

By the method of Lagrange multipliers, the subsidiary condition (b) can be incorporated into the functional J, and we are led to consider the variational equation

(c) $$\delta J' = 0, \qquad \text{(varying } e_{ij}, u_i \text{ independently),}$$

where

(d) $$J' = J + \int_V \lambda_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})]\,dV.$$

Since the quantity in [ ] is symmetric in i, j, we may restrict the Lagrange multipliers λ_{ij} to be symmetric, $\lambda_{ij} = \lambda_{ji}$, so that only six independent multipliers are needed. Derive the Euler equations for (c) and show that λ_{ij} should be interpreted as the stress tensor.

Note: Once Lagrange's multipliers are employed, the phrase "minimum conditions" used in the principle of minimum potential energy has to be replaced by "stationary conditions."

10.8. EXAMPLE OF APPLICATION: STATIC LOADING ON A BEAM—NATURAL AND RIGID END CONDITIONS

As an illustration of the application of the minimum potential energy principle in formulating approximate theories of elasticity, let us consider the approximate theory of bending of a slender beam under static loading. Let the beam be perfectly straight and lying along the x-axis before the application of external loading, which consists of a distributed lateral load $p(x)$ per unit length, a bending moment M_0, and a shearing force Q_0 at the end $x = 0$, and a moment M_l and a shear Q_l at the end $x = l$ (Fig. 10.8:1). We assume that the principal axes of inertia of every cross section of the beam lie in two mutually orthogonal principal planes and that the loading p, M, Q are

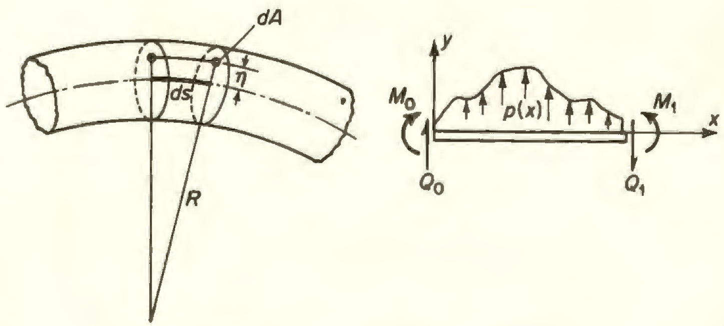


Fig. 10.8:1. Applications to a simple beam.

applied in one of the principal planes. In accordance with the approximate beam theory, we assume that every plane cross section of the beam remains plane during bending.

Let ds be the arc length along the neutral axis. Under these assumptions, we see that when the neutral axis of the beam is bent from the initial straight line into a curve with radius of curvature R, the length of a filament, initially ds and parallel to the neutral axis, is altered by the bending in the ratio $1:(1 + \eta/R)$, where η is the distance between the filament and the neutral axis. The strain is η/R, and the force acting on the filament is $E\,\eta dA/R$, where dA is the cross section of the filament. The resultant moment of these forces about the neutral axis is

$$\int_A \eta \cdot E\frac{\eta}{R}\,dA = \frac{E}{R}\int_A \eta^2\,dA = \frac{EI}{R},$$

where I is the moment of inertia of the area of the beam cross section. The angle through which the cross sections rotate relative to each other is

ds/R. Hence, the work to bend a segment ds of the beam to a curvature $1/R$ is

$$\frac{1}{2}EI\frac{ds}{R^2},$$

where the factor $\frac{1}{2}$ is added since the mean work is half the value of the product of the final moment and angle of rotation. Hence, integrating throughout the beam, we have the strain energy

$$U = \frac{1}{2}\int_0^l EI\frac{ds}{R^2}. \tag{1}$$

If we now assume that the deflection of the beam is *infinitesimal* so that if y denotes the lateral deflection, the curvature $1/R$ is approximated by d^2y/dx^2, and ds is approximated by dx, then

$$U = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx. \tag{1a}$$

The potential energy of the external loading is, with the sign convention specified in Fig. 10.8:1 and with y_0, y_l, $(dy/dx)_0$, $(dy/dx)_l$ denoting the value of y and dy/dx at $x = 0, l$, respectively,

$$-\int p(x)y(x)\,dx + M_0\left(\frac{dy}{dx}\right)_0 - M_l\left(\frac{dy}{dx}\right)_l - Q_0y_0 + Q_ly_l. \tag{2}$$

Hence, the total potential energy is

$$\mathscr{V} = \frac{1}{2}\int_0^l EI\left(\frac{d^2y}{dx^2}\right)^2 dx - \int_0^l py\,dx + M_0\left(\frac{dy}{dx}\right)_0 - M_l\left(\frac{dy}{dx}\right)_l - Q_0y_0 + Q_ly_l. \tag{3}$$

At equilibrium, the variation of $\mathscr{V}$ with respect to the virtual displacement δy must vanish. Hence,

$$\delta\mathscr{V} = \int_0^l EI\frac{d^2y}{dx^2}\,\delta\left(\frac{d^2y}{dx^2}\right)dx - \int_0^l p\,\delta y\,dx + M_0\,\delta\left(\frac{dy}{dx}\right)_0 - M_l\,\delta\left(\frac{dy}{dx}\right)_l - Q_0\,\delta y_0 + Q_l\,\delta y_l = 0.$$

Integrating the first term by parts twice and collecting terms, we obtain

$$\begin{aligned}\delta\mathscr{V} = &\int_0^l \left[\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - p\right]\delta y\,dx \\ &+ \left[EI\left(\frac{d^2y}{dx^2}\right)_l - M_l\right]\delta\left(\frac{dy}{dx}\right)_l - \left[EI\left(\frac{d^2y}{dx^2}\right)_0 - M_0\right]\delta\left(\frac{dy}{dx}\right)_0 \\ &- \left[\frac{d}{dx}EI\left(\frac{d^2y}{dx^2}\right)_l - Q_l\right]\delta y_l + \left[\frac{d}{dx}EI\left(\frac{d^2y}{dx^2}\right)_0 - Q_0\right]dy_0 \\ = &\,0\end{aligned} \tag{4}$$

Since δy is arbitrary in the interval $(0, l)$, we obtain the differential equation of the beam

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) - p = 0, \qquad 0 < x < l. \tag{5}$$

In order that the remaining terms in (4) may vanish, it is sufficient to have the end conditions

(6a) Either $\quad EI\left(\frac{d^2y}{dx^2}\right)_l - M_l = 0 \quad$ or $\quad \delta\left(\frac{dy}{dx}\right)_l = 0.$

(6b) Either $\quad EI\left(\frac{d^2y}{dx^2}\right)_0 - M_0 = 0 \quad$ or $\quad \delta\left(\frac{dy}{dx}\right)_0 = 0.$

(6c) Either $\quad \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)_l - Q_l = 0 \quad$ or $\quad \delta y_l = 0.$

(6d) Either $\quad \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)_0 - Q_0 \quad$ or $\quad \delta y_0 = 0.$

If the deflection y_0 is prescribed at the end $x = 0$, then $\delta y_0 = 0$. If the slope $(dy/dx)_0$ is prescribed at the end $x = 0$, then $\delta(dy/dx)_0 = 0$. These are called "rigid" boundary conditions. On the other hand, if the value of y_0 is unspecified and perfectly free, then δy_0 is arbitrary and we must have

$$\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)_0 - Q_0 = 0 \tag{7}$$

as an end condition for a free end; otherwise $\delta\mathscr{V}$ cannot vanish for arbitrary variations δy_0. Equation (7) is called a "natural" boundary condition. Similarly, all the left-hand equations in (6a)–(6d) are natural boundary conditions, and all the right-hand equations are rigid boundary conditions.

The distinction between natural and rigid boundary conditions assumes great importance in the application of the direct methods of solution of variational problems; the assumed functions in the direct methods must satisfy the rigid boundary conditions. See Sec. 11.8.

It is worthwhile to consider the following question. The reader must be familiar with the fact that in the engineering beam theory, the end conditions often considered are:

(8a) *Clamped end:* $\quad y = 0, \quad \frac{dy}{dx} = 0,$

(8b) *Free end:* $\quad EI\frac{d^2y}{dx^2} = 0, \quad \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) = 0,$

(8c) *Simply supported end:* $\quad y = 0, \quad EI\frac{d^2y}{dx^2} = 0,$

where $y(x)$ is the deflection function of the beam. May we ask why are the other two combinations, namely

$$\text{(9a)} \qquad y = 0, \qquad \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) = 0,$$

$$\text{(9b)} \qquad \frac{dy}{dx} = 0, \qquad EI\frac{d^2y}{dx^2} = 0,$$

never considered?

An acceptable answer is perhaps that the boundary conditions (9a) and (9b) cannot be realized easily in the laboratory. But a more satisfying answer is that they are not proper sets of boundary conditions. If the conditions (9a) or (9b) were imposed, then, according to (4), it can not at all be assured that the equation $\delta\mathscr{V}_i = 0$ will be satisfied. Thus, a basic physical law might be violated. These boundary conditions are, therefore, inadmissible.

From the point of view of the differential Eq. (5), one may feel that the end conditions (9a) or (9b) are legitimate. Nevertheless, they are ruled out by the minimum potential energy principle on physical grounds. In fact, in the theory of differential equations the Eq. (5) and the end conditions (8) are known to form a so-called *self-adjoint* differential system, whereas (5) and (9) would form a *nonself-adjoint* differential system. Very great difference in mathematical character exists between these two catagories. For example, a free vibration problem of a nonself-adjoint system may not have an eigenvector, or it may have complex eigenvalues or complex eigenvectors.

There are other conceivable admissible boundary conditions, such as to require

$$\text{(10)} \qquad \frac{dy}{dx} = 0, \qquad \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) = 0, \qquad \text{at } x = 0.$$

Such an end, with zero slope and zero shear, cannot be easily established in the laboratory. Similarly, it is conceivable that one may require that at the end $x = 0$ the following ratios hold:

$$\text{(11)} \qquad \begin{aligned} &\delta\left(\frac{\partial y}{\partial x}\right) : \delta y = c, \quad \text{a constant,} \\ &\left[\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) - Q_0\right] : \left[EI\frac{d^2y}{dx^2} - M_0\right] = c, \quad \text{the same constant.} \end{aligned}$$

This pair of conditions are also admissible, but are unlikely to be encountered in practice.

10.9. THE COMPLEMENTARY ENERGY THEOREM UNDER SMALL VARIATIONS OF STRESSES

In contrast to the previous sections let us now consider the variation of stresses in order to investigate whether the "actual" stresses satisfy a

minimum principle. We pose the problem as in Sec. 10.7 with a body held in equilibrium under the body force per unit volume F_i and surface tractions per unit area $\overset{\nu}{T}{}_i^*$ over the boundary S_σ, whereas over the boundary S_u the displacements are prescribed. Let σ_{ij} be the "actual" stress field which satisfies the equations of equilibrium and boundary conditions

$$\text{(1)} \qquad \begin{aligned} \sigma_{ij,j} + F_i &= 0 \qquad &\text{in } V, \\ \sigma_{ij}\nu_j &= \overset{\nu}{T}{}_i^* \qquad &\text{on } S_\sigma. \end{aligned}$$

Let us now consider a system of variations of stresses which also satisfy the equations of equilibrium and the stress boundary conditions

$$\text{(2)} \qquad \begin{aligned} (\delta\sigma_{ij})_{,j} + \delta F_i &= 0 \qquad &\text{in } V, \\ (\delta\sigma_{ij})\nu_j &= \delta\overset{\nu}{T}_i \qquad &\text{on } S_\sigma, \\ \delta\sigma_{ij} &\text{ are arbitrary on } S_u. \end{aligned}$$

In contrast to the previous sections, we shall now consider the *complementary virtual work*,

$$\int_V u_i\,\delta F_i\,dv + \int_S u_i\,\delta\overset{\nu}{T}_i\,dS,$$

which, by virtue of (2) and through integration by parts,

$$\begin{aligned} &= -\int_V u_i(\delta\sigma_{ij})_{,j}\,dv + \int_S u_i(\delta\sigma_{ij})\nu_j\,dS \\ &= \int_V (\delta\sigma_{ij})u_{i,j}\,dv - \int_S u_i\nu_j(\delta\sigma_{ij})\,dS + \int_S u_i(\delta\sigma_{ij})\nu_j\,dS \\ &= \tfrac{1}{2}\int_V (\delta\sigma_{ij})(u_{i,j} + u_{j,i})\,dv \\ &= \int_V e_{ij}\,\delta\sigma_{ij}\,dv. \end{aligned}$$

Hence,

$$\text{(3)} \ \blacktriangle \qquad \int_V e_{ij}\,\delta\sigma_{ij}\,dv = \int_V u_i\,\delta F_i\,dv + \int_S u_i\,\delta\overset{\nu}{T}_i\,dS.$$

This equation may be called the *principle of virtual complementary work.* Now, if we introduce *the complementary strain energy* W_c,† which is a function of the stress components $\sigma_{11}, \sigma_{12}, \ldots$, and which has the property that,

$$\text{(4)} \qquad \frac{\partial W_c}{\partial \sigma_{ij}} = e_{ij}$$

† The Gibbs' thermodynamic potential (Sec. 12.3) per unit volume, $\rho\Phi$, is equal to the negative of the complementary strain energy function. If the stress-strain law were linear, then $W_c(\sigma_{ij})$ and $W(e_{ij})$ are equal: $-\rho\Phi = W_c = W$ (linear stress-strain law).

then the complementary virtual work may be written as

$$(5)\qquad \int_V u_i\,\delta F_i\,dv + \int_S u_i\,\delta \overset{\nu}{T}_i\,dS = \int_V \frac{\partial W_c}{\partial \sigma_{ij}}\,\delta\sigma_{ij}\,dv = \delta\int_V W_c\,dv.$$

Since the volume is fixed and u_i are not varied, the result above can be written as

$$(6)\ \blacktriangle \qquad \delta\mathscr{V}^* = 0,$$

where $\mathscr{V}^*$, as a function of the stresses $\sigma_{11}, \sigma_{12}, \ldots$, the surface traction $\overset{\nu}{T}_i$ and the body force per unit volume F_i, is defined as the *complementary energy*

$$(7)\ \blacktriangle \qquad \mathscr{V}^*(\sigma_{11}, \ldots, F_i) \equiv \int_V W_c\,dv - \int_V u_i F_i\,dv - \int_S u_i \overset{\nu}{T}_i\,dS.$$

In practice, we would like to compare stress fields which all satisfy the equations of equilibrium, but not necessarily the conditions of compatibility. In other words, we would have $\delta F_i = 0$ in V and $\delta \overset{\nu}{T}_i = 0$ on S_σ. In this case $\delta\sigma_{ij}$ and, hence, $\delta\overset{\nu}{T}_i$ are arbitrary only on that portion of the boundary where displacements are prescribed, S_u. Therefore, only a surface integral over S_u is left in the left-hand side of (5) and we have

$$(8)\ \blacktriangle \qquad \mathscr{V}^*(\sigma_{11}, \ldots, \sigma_{33}) \equiv \int_V W_c\,dv - \int_{S_u} u_i \overset{\nu}{T}_i\,dS.$$

Therefore, we have the

THEOREM. *Of all stress tensor fields σ_{ij} that satisfy the equation of equilibrium and boundary conditions where stresses are prescribed, the "actual" one is distinguished by a stationary (extreme) value of the complementary energy $\mathscr{V}(\sigma_{11}, \ldots, \sigma_{33})$ as given by* (8).

In this formulation, the linearity of the stress-strain relationship is *not* required, only the existence of the complementary strain energy function is assumed. However, if the stress-strain law were *linear*, and the material is isotropic and obeying Hooke's law, then

$$(9)\qquad W_c(\sigma_{11}, \ldots, \sigma_{33}) = -\frac{\nu}{2E}(\sigma_{\alpha\alpha})^2 + \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij}.$$

We must remark that the variational Eqs. (10.7:2) and Eq. (3) of the present section are applicable even if the body is *not elastic*, for which the energy functional cannot be defined. These variational equations are used in the analysis of inelastic bodies.

Before we proceed further, it may be well worthwhile to consider the concept of complementary work and complementary strain energy. Consider a simple, perfectly elastic bar subjected to a tensile load. Let the relationship between the load P and the elongation of the bar u be given by a unique curve as shown in Fig. 10.9:1. Then the work W is the area between the displacement axis and the curve, while the complementary work W_c is that included between the force axis and the curve. Thus, the two areas complement each other in the rectangular area (force) · (displacement), which would be the work if the force were acting with its full intensity from the beginning of the

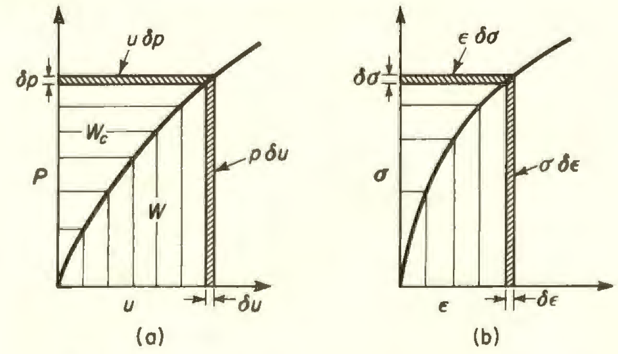


Fig. 10.9:1. Complementary work and strain energy.

displacement. Naturally, W and W_c are equal if the material follows Hooke's law.

The principle of minimum potential energy was formulated by Willard Gibbs; and many beautiful applications were shown by Lord Rayleigh. The complementary energy concept was introduced by F. Z. Engesser; its applications were developed by H. M. Westergaard. In the hands of Kirchhoff, the minimum potential energy theorem becomes the foundation of the approximate theories of plates and shells. Recently, Argyris has made the complementary energy theorem the starting point for practical methods of analysis of complex elastic structures using modern digital computers. See Bibliography 10.2, p. 494.

Let us now return to the complementary energy theorem. We shall show that *in the neighborhood of the natural state, the extreme value of the complementary energy $\mathscr{V}$ is actually a minimum.* By a natural state is meant a state of stable thermodynamic existence (see Sec. 12.4). In the neighborhood of a natural state, the thermodynamic potential per unit volume $\rho\Phi$ can be approximated by a homogeneous quadratic form of the stresses. For an isotropic material, $-\rho\Phi$ is given by Eq. (9), in which W_c is positive definite.

The proof that $\mathscr{V}^*$ is a minimum is analogous to that in the previous section, and it can be sketched as follows. Comparing $\mathscr{V}^*(\sigma_{ij} + \delta\sigma_{ij})$ with $\mathscr{V}^*(\sigma_{ij})$, we have

$$(10)\quad \mathscr{V}^*(\sigma_{11} + \delta\sigma_{11}, \ldots,) - \mathscr{V}^*(\sigma_{11}, \ldots,)$$

$$= \int_V \left[W_c(\sigma_{11} + \delta\sigma_{11}, \ldots,) - W_c(\sigma_{11}, \ldots,) \right] dv$$

$$- \int_{S_u} [\overset{\nu}{T}_i(\sigma_{11} + \delta\sigma_{11}, \ldots,) - \overset{\nu}{T}_i(\sigma_{11}, \ldots,)] u_i \, dS$$

$$= \int_V \left[\left(\frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \theta \, \delta_{ij} \right) \delta\sigma_{ij} + W_c(\delta\sigma_{11}, \ldots,) \right] dv$$

$$- \int_{S_u} (\delta\sigma_{ij}) \nu_j u_i \, dS$$

$$= \int_V e_{ij} \, \delta\sigma_{ij} \, dv - \int_{S_u} (\delta\sigma_{ij}) \nu_j u_i \, dS + \int W_c(\delta\sigma_{11}, \ldots,) \, dv$$

$$= \delta\mathscr{V}^* + \int_V W_c(\delta\sigma_{11}, \delta\sigma_{12}, \ldots,) \, dv,$$

where

$$(11)\quad W_c(\delta\sigma_{11}, \ldots,) = \frac{1+\nu}{2E} (\delta\sigma_{ij})(\delta\sigma_{ij}) - \frac{\nu}{2E} (\delta\sigma_{ii})^2 > 0.$$

$W_c(\delta\sigma_{11}, \ldots,)$ is positive definite for infinitesimal variations $\delta\sigma_{ij}$. Hence, when $\delta\mathscr{V}^* = 0$,

$$(12)\quad \mathscr{V}^*(\sigma_{11} + \delta\sigma_{11}, \ldots,) - \mathscr{V}^*(\sigma_{11}, \ldots,) > 0,$$

and that $\mathscr{V}^*(\sigma_{11}, \ldots,)$ is a minimum is proved.

The converse theorem reads:

Let $\mathscr{V}^*$ be the complementary energy defined by Eq. (8). If the stress tensor field σ_{ij} is such that $\delta\mathscr{V}^* = 0$ for all variations of stresses $\delta\sigma_{ij}$ which satisfy the equations of equilibrium in the body and on the boundary where surface tractions are prescribed, then σ_{ij} also satisfies the equations of compatibility. In other words, the conditions of compatibility are the Euler equation for the variational equation $\delta\mathscr{V}^* = 0$.

The proof was given by Richard V. Southwell[1,2] (1936) through the application of Maxwell and Morera stress functions. We begin with the variational equation

$$(13)\quad \delta\mathscr{V}^* = \int_V e_{ij} \, \delta\sigma_{ij} \, dv - \int_{S_u} u_i \, \delta\overset{\nu}{T}_i \, dS = 0.$$

The variations $\delta\sigma_{ij}$ are subjected to the restrictions

$$(14)\quad (\delta\sigma_{ij})_{,j} = 0 \quad \text{in } V,$$

$$(15)\quad (\delta\sigma_{ij})\nu_j = 0 \quad \text{on } S_\sigma.$$

To accommodate the restrictions (14) into (13), we make use of the celebrated result that the equations of equilibrium (14) are satisfied formally, as can be easily verified, by taking

$$(16)\quad \begin{aligned}
\delta\sigma_{11} &= \phi_{22,33} + \phi_{33,22} - 2\phi_{23,23},\\
\delta\sigma_{22} &= \phi_{33,11} + \phi_{11,33} - 2\phi_{31,31},\\
\delta\sigma_{33} &= \phi_{11,22} + \phi_{22,11} - 2\phi_{12,12},\\
\delta\sigma_{23} &= \phi_{31,12} + \phi_{12,13} - \phi_{11,23} - \phi_{23,11},\\
\delta\sigma_{31} &= \phi_{12,23} + \phi_{23,21} - \phi_{22,31} - \phi_{31,22},\\
\delta\sigma_{12} &= \phi_{23,31} + \phi_{31,32} - \phi_{33,12} - \phi_{12,33},
\end{aligned}$$

where $\phi_{ij} = \phi_{ji}$ are arbitrary stress functions. On setting $\phi_{12} = \phi_{23} = \phi_{31} = 0$, we obtain the solutions proposed by James Clerk Maxwell. On taking $\phi_{11} = \phi_{22} = \phi_{33} = 0$, we obtain the solutions proposed by G. Morera.

Let us use the Maxwell system of arbitrary stress functions for the variations $\delta\sigma_{ij}$. Equation (13) may be written as

$$(17)\quad \begin{aligned}
\delta\mathscr{V}^* = &\int_V [e_{11}(\phi_{22,33} + \phi_{33,22}) + e_{22}(\phi_{33,11} + \phi_{11,33})\\
&+ e_{33}(\phi_{11,22} + \phi_{22,11}) - 2e_{23}\phi_{11,23} - 2e_{31}\phi_{22,31} - 2e_{12}\phi_{33,12}] \, dv\\
&- \int_{S_u} u_i \, \delta\overset{\nu}{T}_i \, dS\\
= &\ 0.
\end{aligned}$$

Integrating by parts twice, we obtain

$$(18)\quad \begin{aligned}
\delta\mathscr{V}^* = &\int_V [(e_{22,33} + e_{33,22} - 2e_{23,23})\phi_{11} + (e_{33,11} + e_{11,33} - 2e_{31,31})\phi_{22}\\
&+ (e_{11,22} + e_{22,11} - 2e_{12,12})\phi_{33}] \, dv + \text{a surface integral}\\
= &\ 0.
\end{aligned}$$

Inasmuch as the stress functions ϕ_{11}, ϕ_{22}, ϕ_{33} are arbitrary in the volume V, the Euler's equations are

$$(19)\quad e_{22,33} + e_{33,22} - 2e_{23,23} = 0,$$

etc., which are Saint-Venant's compatibility equations (see Sec. 4.6). The treatment of the surface integral is cumbersome, but it says only that over

S_u the values of u_i are prescribed and the stresses are arbitrary; the derivation concerns a certain relationship between ϕ_{ij} and their derivatives to be satisfied on the boundary.

Similarly, the use of Morera system of arbitrary stress functions leads to the other set of Saint-Venant's compatibility equations

(20) $$e_{11,23} = -e_{23,11} + e_{31,21} + e_{12,31},$$

etc. [Eq. (4.6:4)].

If we start with $\delta\mathscr{V}^*$ in the form

(21) $$\delta\mathscr{V}^* = 0 = \int_V \left(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{\alpha\alpha}\delta_{ij}\right)\delta\sigma_{ij}\,dv + \int_{S_u} u_i\,\delta\overset{\nu}{T}_i\,dS$$

and introduce the stress functions, the Beltrami-Michell compatibility equations

(22) $$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\theta_{,ij} + \frac{\nu}{1-\nu}\delta_{ij}F_{k,k} + F_{i,j} + F_{j,i} = 0 \qquad \text{in } V$$

can be obtained directly.

In concluding this section, we remark once more that when we consider the variation of the stress field of a body in equilibrium, the principle of virtual complementary work has broad applicability. The introduction of the complementary energy functional $\mathscr{V}^*$, however, limits the principle to elastic bodies. That $\mathscr{V}^*$ actually is a minimum with respect to all admissible variations of stress field is established only if the complementary strain energy function is positive definite.

There are many fascinating applications of the minimum complementary energy principle. In Sec. 10.11 below we shall consider its application in proving Saint-Venant's principle in Zanaboni's formulation.

Problem 10.2. The principle of virtual complementary work states that

(23) $$\int_V e_{ij}\delta\sigma_{ij}\,dv - \int_{S_u} u_i^*\delta\overset{\nu}{T}_i\,dS = 0$$

under the restrictions that

(24) $$\delta\sigma_{ij,j} = 0 \quad \text{in } V,$$

(25) $$\delta\sigma_{ij}\nu_j = 0 \quad \text{on } S_\sigma.$$

(26) $$u_i = u_i^* \quad \text{prescribed on } S_u, \text{ but } \delta\overset{\nu}{T}_i = \delta\sigma_{ij}\nu_j \text{ arbitrary on } S_u.$$

Using Lagrange multipliers, we may restate this principle as

(27) $$\int_V e_{ij}\delta\sigma_{ij}\,dv - \int_{S_u} u_i^*\delta\overset{\nu}{T}_i\,dS + \int_V \lambda_i\delta\sigma_{ij,j}\,dv - \int_{S_\sigma} \mu_i\delta\sigma_{ij}\nu_j\,dS = 0$$

The six equations (24), (25), $i = 1, 2, 3$, require six Lagrange multipliers λ_i, μ_i which are functions of (x_1, x_2, x_3). Show that the Euler equations for (27) yield the physical interpretation

(28) $$\lambda_i = u_i, \qquad \mu_i = u_i.$$

10.10. REISSNER'S PRINCIPLE

Consider the functional $J(e_{ij}, u_k, \sigma_{ij})$, where $\sigma_{ij} = \sigma_{ji}$:

(1) $$J = \int_V [W(e_{ij}) - F_iu_i]\,dv - \int_V \sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})]\,dv$$
$$- \int_{S_\sigma} \overset{\nu}{T}_i{}^*u_i\,dS - \int_{S_u} \sigma_{ij}\nu_j(u_i - u_i^*)\,dS, \qquad (S = S_\sigma + S_u).$$

Let us seek the necessary conditions for J to be stationary. On setting the first variation of J to zero, permitting the fifteen functions e_{ij}, u_i, and σ_{ij}, $(i = 1, 2, 3)$ to vary over the domain V, and u_i to vary over S_σ, and σ_{ij} to vary over S_u, while F_i, $\overset{\nu}{T}_i^*$, and u_i^* are prescribed, we obtain

(2) $$0 = \delta J = \int_V \left\{\frac{\partial W}{\partial e_{ij}}\delta e_{ij} - F_i\,\delta u_i - \delta\sigma_{ij}[e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})]\right.$$
$$\left. - \sigma_{ij}[\delta e_{ij} - \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i})]\right\} dv$$
$$- \int_{S_\sigma} \overset{\nu}{T}_i{}^*\,\delta u_i\,dS - \int_{S_u} [\delta\sigma_{ij}\nu_j(u_i - u_i^*) + \sigma_{ij}\nu_j\,\delta u_i]\,dS.$$

Integrating by parts those terms involving $\delta u_{i,j}$, we obtain the Euler equations:

(3) $$\frac{\partial W}{\partial e_{ij}} = \sigma_{ij} \quad \text{in } V,$$

(4) $$\sigma_{ij,j} + F_i = 0 \quad \text{in } V,$$

(5) $$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V,$$

(6) $$\sigma_{ij}\nu_j = \overset{\nu}{T}_i{}^* \quad \text{on } S_\sigma,$$

(7) $$u_i = u_i^*, \qquad \delta u_i = 0 \quad \text{on } S_u.$$

Clearly these are the basic equations of linear elasticity (Section 7.1).

The Eq. (2), $\delta J = 0$, where J is defined in (1), is a statement of Reissner's principle as applied to linear elasticity.

The motivation for J is clear. The functional

$$J_1 = \int_V \{W(e_{ij}) - F_iu_i\}\,dv - \int_{S_\sigma} \overset{\nu}{T}_i{}^*u_i\,dS$$

represents the sum of the strain energy and the potential energy of the external loads. Now if we wish to find a stationary value of J_1 by varying e_{ij} and u_i, subjected to the subsidiary conditions (5), (6), and (7), we may do so by introducing Lagrange multipliers. Obviously σ_{ij} plays the rôle of the Lagrangian multiplier in (1).

Eric Reissner,[10.2] announced this principle in 1950 and extended it to finite elastic deformations in 1953. In Reissner's original paper, the complementary energy functional was used instead of the potential energy. In the infinitesimal displacement theory, the complementary energy is defined as

$$W_c(\sigma_{ij}) = \sigma_{ij}e_{ij} - W(e_{ij}). \tag{8}$$

$W_c(\sigma_{ij})$ is a function of the stress components if e_{ij} on the right-hand side are expressed in terms of σ_{kl} by means of Hooke's law. $W_c(\sigma_{ij})$ has the property that

$$\frac{\partial W_c}{\partial \sigma_{ij}} = e_{ij}. \tag{9}$$

On substituting $W = \sigma_{ij}e_{ij} - W_c$ from Eq. (8) into Eq. (1), we have Reissner's functional J_R:

$$J_R = \int_V \{-W_c(\sigma_{ij}) - F_i u_i + \tfrac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i})\}\, dv - \int_{S_\sigma} \overset{\nu}{T}{}_i^{*} u_i\, dS - \int_{S_u} \sigma_{ij}\nu_j(u_i - u_i^{*})\, dS \tag{10}$$

and Reissner's principle that the elastic equilibrium is distinguished by $\delta J_R = 0$, when σ_{ij} and u_i $(i, j = 1, 2, 3)$ are varied independently, provided that the tensor σ_{ij} is symmetric.

If we vary J_R with respect to σ_{ij} only, $(\delta u_i \equiv 0)$, keeping the stress conditions (4) and (6) satisfied, then the complementary energy theorem is obtained.

10.11. SAINT-VENANT'S PRINCIPLE

In 1855, Barre de Saint-Venant enunciated the "principle of the elastic equivalence of statically equipollent systems of loads." According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part. When this principle is applied to the problem of torsion of a long shaft due to a couple applied at its ends, it states that the shear stress distribution in the shaft at a distance from the ends large compared with the cross-sectional dimension of the shaft will be practically independent of the exact distribution of the surface tractions of

which the couple is the resultant. Such a principle is nearly always applied, consciously or unconsciously, when we try to simplify or idealize a problem in mathematical physics. It is used, for example, in devising a simple tension test for a material, when we clamp the ends of a test specimen in the jaws of a testing machine and assume that the action on the central part of the bar is nearly the same as if the forces were uniformly applied at the ends.

The justification of the principle is largely empirical and, as such, its interpretation is not entirely clear.

One possible way to formulate Saint-Venant's principle with mathematical precision is to state the principle in certain sense of average as follows (Zanaboni,[10.3] 1937, Locatelli,[10.3] 1940, 1941). Consider a body as shown in Fig. 10.11:1. A system of forces P in static equilibrium (with zero resultant force and zero resultant couple) is applied to a region of the body enclosed in a small sphere B. Otherwise the body is free. Let S' and S'' be

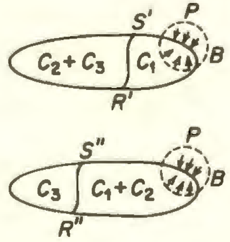


Fig. 10.11:1

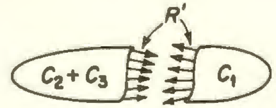


Fig. 10.11:2

two arbitrary nonintersecting cross sections, both outside of B, with S'' farther away from B than S'. Due to the system of loads P, stresses are induced in the body. If we know these stresses, we can calculate the tractions acting on the surfaces S' and S''. Let the body be considered as severed into two parts at S', and let the system of surface tractions acting on the surface S' be denoted by R' which is surely a system of forces in equilibrium (see Fig. 10.11:2). Then a convenient measure of the magnitude of the tractions R' is the total strain energy that would be induced in the two parts should they be loaded by R' alone. Let this strain energy be denoted by $U_{R'}$. We have,

$$U_{R'} = \int_V W(\sigma_{ij}^{(R)})\, dv,$$

where W is the strain energy density function, and the stresses $\sigma_{ij}^{(R)}$ correspond to the loading system R'. Similarly, let the magnitude of the stresses at the section S'' be measured by the strain energy $U_{R''}$, which would have been induced by the tractions R'' acting on the surface S'' over the two parts of the body severed at S''. Both $U_{R'}$ and $U_{R''}$ are positive quantities and they vanish only if R', R'' vanish identically. Now we shall formulate Saint-Venant's principle in the following form (Zanaboni,[10.3] 1937).

Let S' and S'' be two nonintersecting sections both outside a sphere B. If the section S'' lies at a greater distance than the section S' from the sphere B in which a system of self-equilibrating forces P acts on the body, then

$$(1) \qquad U_{R''} < U_{R'}.$$

In this form, the diminishing influence of the self-equilibrating system of loading P as the distance from B increases is expressed by the functional U_R, which is a special measure of the stresses induced at any section outside B. The reason for the choice of U_R as a measure is its positive definiteness character and the simplicity with which the theorem can be proved. Further sharpening of the principle will be discussed later.

In order to prove the theorem (1), we first derive an auxiliary principle. Let a self-equilibrating system of forces P be applied to a limited region B at the surface of an otherwise free elastic body C_1, (Fig. 10.11:3). Let U_1 be the strain energy produced by P in C_1. Let us now consider an enlarged body $C_1 + C_2$ by affixing to C_1 an additional body C_2 across a surface S which does not intersect the region B. When P is applied to the enlarged body $C_1 + C_2$, the strain energy induced is denoted by U_{1+2}. Then the lemma states that

$$(2) \qquad U_{1+2} < U_1.$$

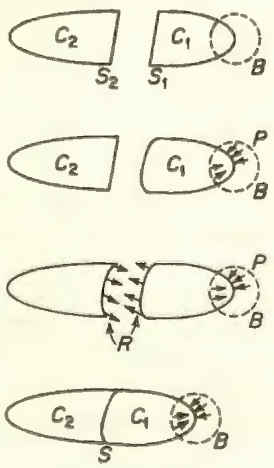


Fig. 10.11:3

Proof of the Lemma. To compute U_{1+2}, we imagine that the stresses in the enlarged body $C_1 + C_2$ are built up by the following steps (see Fig. 10.11:3). First the load P is applied to C_1. The face S_1 of C_1 is deformed. Next a system of surface tractions R is applied to C_1 and C_2 on the surfaces of separation, S_1 and S_2. R will be so chosen that the deformed surfaces S_1 and S_2 fit each other exactly, so that displacements of material points in C_1 and C_2 are continuous as well as the stresses. Now C_1 and C_2 can be brought together and welded, and S becomes merely an interface. The result is the same if C_1 and C_2 were joined in the unloaded state and the combined body $C_1 + C_2$ is loaded by P.

The strain energy U_{1+2} is the sum of the work done by the forces in the above stages. In the first stage, the work done by P is U_1. In the second stage, the work done by R on C_2 is U^*_{R2}; the work done by R on C_1 consists of two parts, U^*_{R1} if C_1 were free, and U^*_{PR}, the work done by the system of loads P due to the deformation caused by R. Hence,

$$(3) \qquad U_{1+2} = U_1 + U^*_{R1} + U^*_{R2} + U^*_{PR}.$$

Now the system of forces R represents the internal normal and shear stresses acting on the interface S of the body $C_1 + C_2$. It is therefore determined by the minimum complementary energy theorem. Consider a special variation of stresses in which all the actual forces R are varied in the ratio $1:(1+\epsilon)$, where ϵ may be positive or negative. The work U^*_{R1} will be changed to $(1+\epsilon)^2 U^*_{R1}$, because the load and deformation will both be changed by a factor $(1+\epsilon)$. Similarly, U^*_{R2} is changed to $(1+\epsilon)^2 U^*_{R2}$. But U^*_{PR} is only changed to $(1+\epsilon)U^*_{PR}$, because the load P is fixed, while the deformation is varied by a factor $(1+\epsilon)$. Hence, U_{1+2} is changed to

$$(4) \qquad U'_{1+2} = U_1 + (1+\epsilon)^2 U^*_{R1} + (1+\epsilon)^2 U^*_{R2} + (1+\epsilon) U^*_{PR}.$$

The difference between (4) and (3) is

$$\Delta U_{1+2} = \epsilon(2U^*_{R1} + 2U^*_{R2} + U^*_{PR}) + \epsilon^2(U^*_{R1} + U^*_{R2}).$$

For U_{1+2} to be a minimum, ΔU_{1+2} must be positive regardless of the sign of ϵ. This is satisfied if

$$(5) \qquad 2U^*_{R1} + 2U^*_{R2} + U^*_{PR} = 0.$$

On substituting (5) into (3), we obtain

$$(6) \qquad U_{1+2} = U_1 - (U^*_{R1} + U^*_{R2}).$$

Since U^*_{R1} and U^*_{R2} are positive definite, we see that lemma (2) is proved.

Now we shall prove Saint-Venant's principle embodied in Eq. (1). Consider an elastic body consisting of three parts $C_1 + C_2 + C_3$ loaded by P in B, as shown in Fig. 10.11:1. Let this body be regarded first as a result of adjoining $C_2 + C_3$ to C_1 with an interface force system R', and then as a result of adjoining C_3 to $C_1 + C_2$ with an interface force R''. We have, by repeated use of (6),

$$\begin{aligned} U_{1+(2+3)} &= U_1 - (U^*_{R'1} + U^*_{R'(2+3)}), \\ U_{(1+2)+3} &= U_{1+2} - (U^*_{R''(1+2)} + U^*_{R''3}) \\ &= U_1 - (U^*_{R1} + U^*_{R2}) - (U^*_{R''(1+2)} + U^*_{R''3}). \end{aligned}$$

Equating these expressions, we obtain

$$U^*_{R'1} + U^*_{R'(2+3)} = U^*_{R1} + U^*_{R2} + U^*_{R''(1+2)} + U^*_{R''3},$$

or, since U^*_{R1} and U^*_{R2} are essentially positive quantities,

$$(7) \qquad U^*_{R'1} + U^*_{R'(2+3)} > U^*_{R''(1+2)} + U^*_{R''3}.$$

This is Eq. (1), on writing $U_{R'}$ for $U^*_{R'1} + U^*_{R'(2+3)}$, etc. Hence, the principle is proved.

10.12. SAINT-VENANT'S PRINCIPLE—BOUSSINESQ-VON MISES-STERNBERG FORMULATION

The Saint-Venant principle, as enunciated in terms of the strain energy functional, does not yield any detailed information about individual stress components at any specific point in an elastic body. However, such information is clearly desired. To sharpen the principle, it may be stated as follows (von Mises, 1945). "If the forces acting upon a body are restriced to several small parts of the surface, each included in a sphere of radius ϵ, then the strains and stresses produced in the interior of the body at a finite distance from all those parts are smaller in order of magnitude when the forces for each single part are in equilibrium than when they are not."

The classical demonstration of this principle is due to Boussinesq (1885), who considered an infinite body filling the half-space $z > 0$ and subjected to several concentrated forces, each of magnitude F, normal to the boundary $z = 0$. If these normal forces are applied to points in a small circle B with diameter ϵ, Boussinesq proved that the largest stress component at a point P which lies at a distance R from B is

(1) of order F/R^2 if the resultant of the forces is of order F,
(2) of order $(\epsilon/R)(F/R^2)$ if the resultant of the forces is zero,
(3) of order $(\epsilon/R)^2(F/R^2)$ if both the resultant force and the resultant moment vanish.

(Cf. Secs. 8.8, 8.10.)

These relative orders of magnitude were believed to have general validity until von Mises (1945) showed that a modification is necessary.

Consider the half-space $z > 0$ again. Let forces of magnitude F *tangent* to the boundary $z = 0$ be applied to points in a small circle B of diameter ϵ. Making use of the well-known Cerruti solution (Sec. 8.8), von Mises obtained

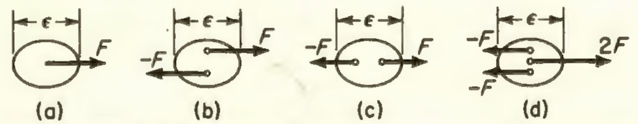


Fig. 10.12:1. von Mises' examples in which forces tangential to the surface of an elastic half-space are applied in a small area.

the following results for the four cases illustrated in Fig. 10.12:1. The order of magnitude of the largest stress component at a point P which lies at a distance R from B is

(1) of order $\sigma_0 = F/R^2$ in case (a),
(2) of order $(\epsilon/R)\sigma_0$ in case (b),
(3) of order $(\epsilon/R)\sigma_0$ in case (c),
(4) of order $(\epsilon/R)^2\sigma_0$ in case (d).

The noteworthy case is (c), which is drastically different from what one would expect from an indiscriminating generalization of Boussinesq's result, for in this case the forces are in static equilibrium, with zero moment about any axes, but all one could expect is a stress magnitude of order no greater than $(\epsilon/R)\sigma_0$, not $(\epsilon/R)^2\sigma_0$. von Mises found that in this case the order of magnitude of the largest stress component is reduced to $(\epsilon/R)^2\sigma_0$ if and only if the external forces acting upon a small part of the surface are such as to remain in equilibrium when all the forces are turned through an arbitrary angle. (Such a case is called *astatic equilibrium.*)

von Mises examined next the stresses in a finite circular disk due to loads acting on the circumference, and a similar conclusion was reached. (See Fig. 10.12:2.)

These examples show that Saint-Venant's principle, as stated in the traditional form at the beginning of this section, does not hold true.

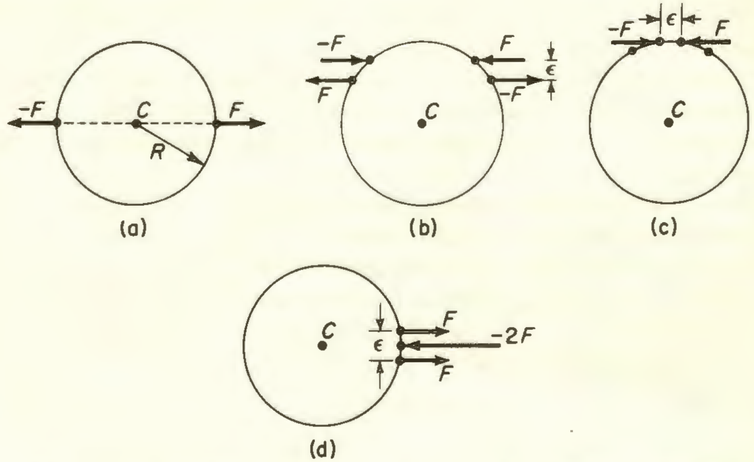


Fig. 10.12:2. von Mises' examples of a circular disk subjected to loads on the circumference.

Accordingly, von Mises[10.3] proposed, and later Sternberg[10.3] proved (1954), the following mathematical statement of the Saint-Venant principle:

Let a body be acted on by surface tractions which are finite and are distributed over a number of regions all no greater than a sphere of diameter ϵ. Consider an interior point x whose distance to any of these loading areas in no less than a characteristic length which will be taken as unity. ϵ is nondimensionalized with respect to this characteristic length. Then, as $\epsilon \to 0$, the order of magnitude of the strain components at x is as follows:

$$e(x, \epsilon) = O(\epsilon^p)$$

where

(a) If the tractions have nonvanishing vector sums in at least one area, then in general, $\rho > 2$. (Note that the surface traction is assumed to be finite, so the resultant force $\rightarrow 0$ as ϵ^3, since the area on which the surface tractions act $\rightarrow 0$ as ϵ^2.)

(b) If the resultant of the surface tractions in every loading area vanishes, then $\rho > 3$.

(c) If, in addition, the resultant moment of the surface tractions in every loading area also vanishes, then still we can be assured only of $\rho > 3$.

(d) $\rho > 4$ in case of astatic equilibrium in every loading area, which may be described by the 12 scalar conditions

$$\int_{S(\epsilon)} T_i^* \, dS = 0, \qquad \int_{S(\epsilon)} T_i^* x_j \, dS = 0, \qquad (i, j = 1, 2, 3)$$

for each loading area $S(\epsilon)$, where T_i^* represents the specified surface traction over $S(\epsilon)$.

If the tractions applied to $S(\epsilon)$ are parallel to each other and not tangential to the surface, then if they are in equilibrium they are also in astatic equilibrium, and the condition $\rho > 4$ prevails.

If, instead of prescribing finite surface traction, we consider finite forces being applied which remain finite as $\epsilon \rightarrow 0$, then the exponent ρ should be replaced by 0, 1, 2 in the cases named above, as was illustrated in Figs. 10.12:1 and 10.12:2.

The theorem enunciated above does not preclude the validity of a stronger Saint-Venant principle for special classes of bodies, such as thin plates or shells or long rods. With respect to perturbations that occur at the edges of a thin plate or thin shell, a significant result was obtained by K. O. Friedrichs[10.3] (1950) in the form of a so-called boundary-layer theory. With respect to lateral loads on shells, Naghdi[10.3] (1960) obtained results similar to von Mises-Sternberg's.

10.13. PRACTICAL APPLICATIONS OF SAINT-VENANT'S PRINCIPLE

It is well-known that Saint-Venant's principle has its analogy in hydrodynamics and that these features are associated with the elliptic nature of the partial differential equations. If the differential equations were hyperbolic and two-dimensional, local disturbances may be propagated far along the characteristics without attenuation. Then the concept of the Saint-Venant's principle will not apply. For example, in the problem of the response of an elastic half-space to a line load traveling at supersonic speeds over the free surface, as discussed in Sec. 9.7, the governing equations (9.7:11) are hyperbolic, and we know that any fine structure of the surface pressure distribution is propagated all the way to infinity.

On the other hand, one feels intuitively that the validity of Saint-Venant's

principle is not limited to linear elastic solid or infinitesimal displacements. One expects it to apply in the case of rubber for finite strain or to steel even when yielding occurs. Although no precise proof is available, Goodier[10.3] (1937) has argued on the basis of energy as follows:

Let a solid body be loaded in a small area whose linear dimensions are of order ϵ, with tractions which combine to give zero resultant force and couple. Such a system of tractions imparts energy to the solid through the *relative* displacements of the points in the small loaded area, because no work is done by the tractions in any translation or rotation of the area as a rigid body. Let the tractions be of order p. Let the slope of the stress strain curve of the material be of order E. (The stress-strain relationship does not have to be linear.) Let one element of the loaded area be regarded as fixed in position and orientation. Then since the strain must be of order p/E, the displacements of points within the area are of order $(p\epsilon/E)$. The work done by the traction acting on an element dS is of order $(p\epsilon p \cdot dS/E)$. The order of magnitude of total work is, therefore, $p^2\epsilon^3/E$. Since a stress of order p implies a strain energy of order p^2/E *per unit volume*, the region in which the stress is of order p must have a volume comparable with ϵ^3. Hence, the influence of tractions cannot be appreciable at a distance from the loaded area which are large compared with ϵ.

Goodier's argument can be extended to bodies subjected to limited plastic deformation. In fact, the argument provides an insight to practical judgement of how local self-equilibrating tractions should influence the strain and stress in the interior of a body.

An engineer needs to know not only the order of magnitude comparison such as stated in von Mises-Sternberg theorem; he needs to know also how numerically trustworthy Saint-Venant's principle is to his particular problem. Hoff[10.3] (1945) has considered several interesting examples, two of which are given in Figs. 10.13:1 and 10.13:2 and will be explained below.

In the first example, the torsion of beams with different cross sections is considered. One end of the beam is clamped, where cross sectional warping is prevented. The other end is free, where a torque is applied by means of shear stresses distributed according to the requirements of the theory of pure torsion. The difference between the prescribed end conditions at the clamped end from those assumed in Saint-Venant's torsion theory (Sec. 7.5) may be stated in terms of a system of self-equilibrating tractions that act at the clamped end. Timoshenko has given approximate solutions to these problems. Hoff's example refers to beams with dimensions as shown in the figure and subjected to a torque of 10 in. lb. Due to the restrictions of warping, normal stresses are introduced in the bar in addition to the shearing stresses of Saint-Venant's torsion. For the rectangular beam, the maximum normal stress at the fixed end is equal to 157 lb/sq in. For the other two thin-walled channel sections, the maximum normal stresses at the fixed end are 1230 and 10,900 lb/sq in., respectively, for the thicker and thinner sections. In Fig.

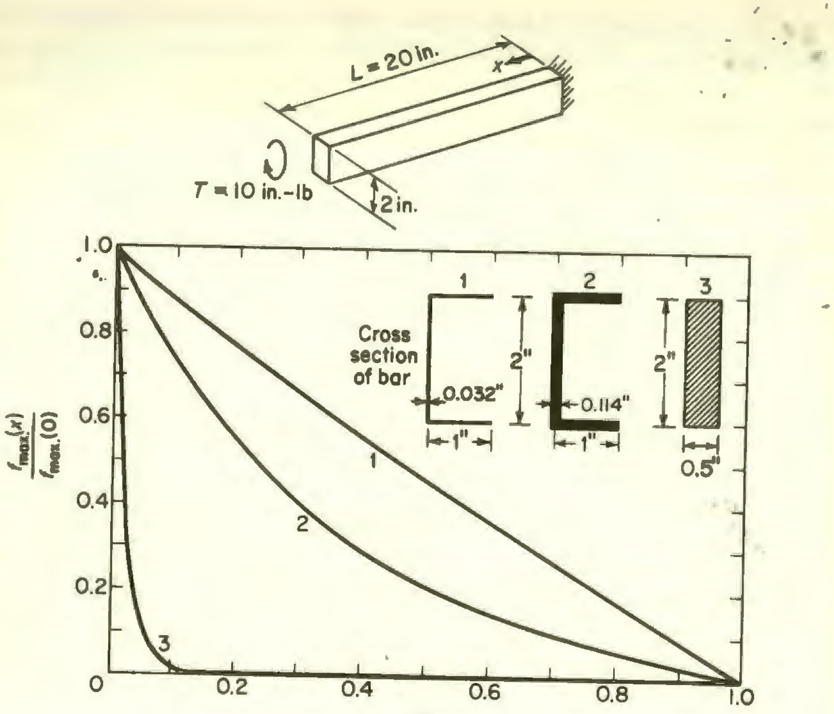


Fig. 10.13:1. Hoff's illustration of St-Venant's principle.

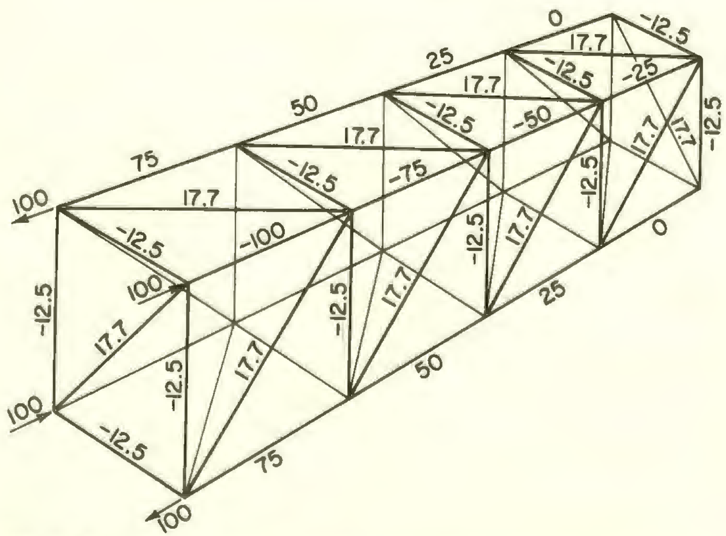


Fig. 10.13:2. Hoff's example illustrating the slow decay of self-equilibrating forces in a space framework.

10.13:1 curves are shown for the ratio of maximum normal stress $f_{max}(x)$ in any section x divided by the maximum normal stress $f_{max}(0)$ in the fixed-end section, plotted against the ratio distance x of the section from the fixed end of the bar, divided by the total length L of the bar. Inspection of the curves reveals that, while in the case of the solid rectangular section the normal stress caused by the restriction to warping at the fixed end is highly localized, it has appreciable values over the entire length of the channel section bars. Consequently, reliance on Saint-Venant's principle in the calculation of stresses caused by torsion is entirely justified with the bar of rectangular section. In contrast, stresses in the thin-walled section bars depend largely upon the end conditions.

Hoff's second example refers to pin-jointed space frameworks. In one case a statically determinate framework is considered, to one end of which a set of four self-equilibrating concentrated loads are applied, as shown in Fig. 10.13:2. The figures written on the elements of the framework represent the forces acting in the bars measured in the same units as the applied loads. Negative sign indicates compression. It can be seen that the effect of the forces at one end of the structure is still noticeable at the other end.

Hoff's examples show that Saint-Venant's principle works only if there is a possibility for it to work; in other words, only if there exist paths for the internal forces to follow in order to balance one another within a short distance of the region at which a group of self-equilibrating external forces is applied. This point of view is in agreement with Goodier's reasoning.

Hoff's examples are not in conflict with the von Mises-Sternberg theorem, for the latter merely asserts a certain order of magnitude comparison for the stress and strain as the size of the region of self-equilibrating loading shrinks to zero, and does not state how the stresses are propagated. On the other hand, although Goodier's reasoning does not provide a definitive theorem, it is very suggestive in pointing out the basic reason for Saint-Venant's principle and can be used in estimating the practical efficiency of the principle.

PROBLEMS

10.3. Derive the differential equation and permissible boundary conditions for a membrane stretched over a simply connected regular region A by minimizing the functional I with respect to the displacement w; $p(x, y)$ being a given function

$$I = \frac{1}{2}\iint_A (w_x^2 + w_y^2)\,dx\,dy - \iint_A pw\,dx\,dy.$$

10.4. Obtain Euler's equation for the functional

$$I' = \frac{1}{2}\iint_A [w_x^2 + w_y^2 + (\nabla^2 w + p)^2 - 2pw]\,dx\,dy.$$

Ans. $\nabla^4 w - \nabla^2 w + \nabla^2 p - p = 0.$

Note: Compare I' with I of Prob. 10.3. Since $\nabla^2 w + p = 0$ for a membrane, the solution of Prob. 10.3 also satisfies the present problem. But the integrand of I' contains higher derivatives of w and the equation $\delta I' = 0$ is a sharper condition than the equation $\delta I = 0$. These examples illustrate Courant's method of sharpening a variational problem and accelerating the convergence of an approximating sequence in the direct method of solution of variational problems.

10.5. Bateman's principle in fluid mechanics states that in a flow of an ideal, nonviscous fluid (compressible or incompressible) in the absence of external body forces, the "pressure integral"

$$-\iiint p\, dx\, dy\, dz,$$

which is the integral of pressure over the entire fluid domain, is an extremum. Consider the special case of a steady, irrotational flow of an incompressible fluid, for which the Bernoulli's equation gives

$$p = \text{const.} - \frac{\rho V^2}{2} = \text{const.} - \frac{\rho}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right],$$

where ϕ is the velocity potential. Derive the field equation governing ϕ and the corresponding natural boundary conditions according to Bateman's principle. [H. Bateman, *Proc. Roy. Soc. London*, **A, 125** (1929) 598–618.]

10.6. Let

$$T = \sum_{\mu,\nu=1}^{n} P_{\mu\nu}(t, q_1, \ldots, q_m)\, \dot q_\mu(t)\, \dot q_\nu(t),$$

$$U = U(t, q_1, \ldots, q_m),$$

and

$$L = T - U,$$

where $\dot q_\mu(t) = dq_\mu/dt$, and $q_\mu(t_0) = a_\mu$, $q_\mu(t_1) = b_\mu$, ($\mu = 1, \ldots, m$); a_μ, b_μ are given numbers. Derive the Euler differential equations for the variational problem connected with the functional

$$J[q_1, \ldots, q_m] = \int_{t_0}^{t_1} L\, dt.$$

Prove that

$$\frac{d(T + U)}{dt} = 0,$$

if $\partial U/\partial t = 0$, $\partial P_{\mu\nu}/\partial t = 0$, $\mu, \nu = 1, \ldots, m$. *Hint:* Use the Euler's relation for homogeneous functions.

10.7. Find the curves in the x, y-plane such that

$$\int_a^b \sqrt{2E - n^2 y^2}\, ds, \qquad ds^2 = dx^2 + dy^2$$

is stationary, where E and n are constants and the integral is taken between fixed end points.

10.8. Let D be the set of all functions $u(x)$ with the following properties:

$$D: \begin{cases} (1) & u(x) = a_1 \sin \pi x + a_2 \sin 2\pi x, \\ (2) & a_1, a_2 \text{ are real, arbitrary numbers,} \\ (3) & a_1^2 + a_2^2 > 0. \end{cases}$$

Consider the functional

$$J[u] = \frac{\int_0^1 [u'(x)]^2\, dx}{\int_0^1 [u(x)]^2\, dx}$$

for all functions $u(x)$ in D.

Questions:

(a) Give necessary conditions for a function $u^*(x)$ in D so that $u^*(x)$ furnishes an extremum of the functional J.

(b) Show that there exist exactly two functions

$$u_1(x) = a_{11} \sin \pi x + a_{12} \sin 2\pi x,$$
$$u_2(x) = a_{21} \sin \pi x + a_{22} \sin 2\pi x,$$

notwithstanding a constant factor, which satisfy these necessary conditions mentioned in (a).

(c) Show that for one of the functions mentioned in (b), say, $u_1(x)$, the inequality

$$J[u_1] < J[u]$$

holds for all functions $u(x)$ in D. Then show that for the other solution, $u_2(x)$, the inequality

$$J[u] < J[u_2]$$

holds for all functions $u(x)$ in D.

(d) Does there exist any relation between $J[u_1]$, $J[u_2]$ and the eigenvalues of the eigenvalue problem

$$-u''(x) = \lambda u(x), \qquad 0 < x < 1,$$

with

$$u(0) = u(1) = 0.$$

10.9. Clapeyron's theorem states that, if a linear elastic body is in equilibrium under a given system of body forces F_i and surface forces $\overset{\nu}{T}_i$, then the strain energy of deformation is equal to one-half the work that would be done by the external forces (of the equilibrium state) acting through the displacements u_i from the unstressed state to the state of equilibrium; i.e.,

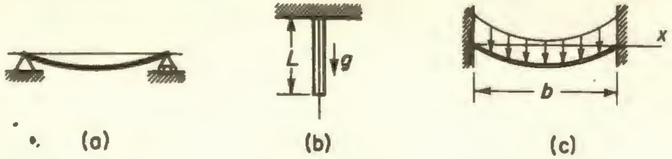
$$\int_V F_i u_i\, dv + \int_S \overset{\nu}{T}_i u_i\, dS = 2\int_V W\, dv.$$

Demonstrate Clapeyron's Theorem for:

(a) A simply supported beam under its own weight [Fig. P10.9(a)].

(b) A rod under its own weight [Fig. P10.9(b)],

$$u(x) = \frac{g\rho L^2}{E}\left[\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right].$$

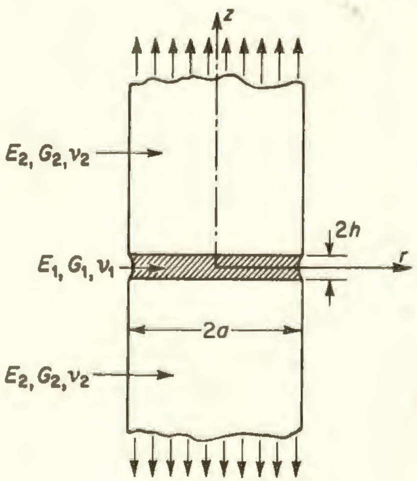


P10.9.

(c) A strip of membrane of infinite length and width b [Fig. P10.9(c)], under a constant pressure p_0, for which

$$w = w_0(1 - x^2), \qquad \frac{w_0}{h} = \sqrt[3]{\frac{3(1 - \nu^2)}{64}\frac{p_0 b^4}{Eh^4}}.$$

10.10. *The "poker chip" problem* (*Max Williams*). To obtain a nearly triaxial tension test environment for polymer materials, a "poker chip" test specimen (a short circular cylinder) is glued between two circular cylinders (Fig. P10.10). When the cylinder is subjected to simple tension, the center of the poker chip is subjected to a triaxial tension stress field. Let the elastic constants of the media be E_1, G_1, ν_1 and E_2, G_2, ν_2, as indicated in Fig. P10.10, with $E_1 \ll E_2$, $\nu_1 \cong 0.5$. Assume cylindrical symmetry, and obtain approximate expressions of the stress field by the following methods.



P10.10 Poker chip specimen.

(a) Use the complementary energy theorem and a stress field satisfying stress boundary conditions and the equations of equilibrium.

(b) Use the potential energy theorem and assumed displacements satisfying displacement boundary conditions.

(c) Experimental results suggest that the following displacement field is reasonable:

$$w = w_0\frac{z}{h}, \qquad -h < z < h,$$

$$u = \left(1 - \frac{z^2}{h^2}\right)g(r),$$

where $g(r)$ is as yet an unknown function of r. Obtain the governing equation for $g(r)$ and its appropriate solution for this problem by attempting to satisfy the following equilibrium equations in some average sense with respect to the z-direction:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0,$$

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0.$$

10.11. Consider a pin-ended column of length L and bending stiffness EI subjected to an end thrust P. A spring is attached at the middle of the column as shown in Fig. P10.11. When the column is straight the spring tension is zero. If the column deflects by an amount Δ, the spring exerts a force R on the column:

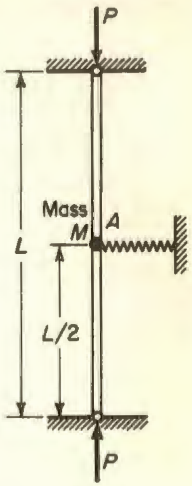
$$R = K\Delta - \alpha\Delta^3, \qquad (K > 0,\ \alpha > 0).$$

Derive the equation of equilibrium of the system. It is permissible to use the Euler-Bernoulli approximation for a beam, for which the strain energy per unit length is

$$\tfrac{1}{2}EI \cdot (\text{curvature})^2$$

and

$$\text{bending moment} = EI \cdot (\text{curvature}).$$



P10.11

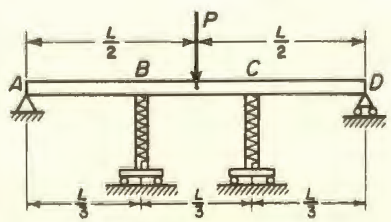
10.12. For the column of Prob. 10.11 under the axial load P, is the solution unique? Under what situation is the solution nonunique? What are the possible solutions when the uniqueness is lost?

10.13. A linear elastic beam of bending stiffness EI is supported at four equidistant points $ABCD$, Fig. P10.13. The supports at A and D are pin-ended. At B and C, the beam rests on two identical nonlinear pillars. The characteristic of the pillars may be described as "hardening" and is expressed by the equation

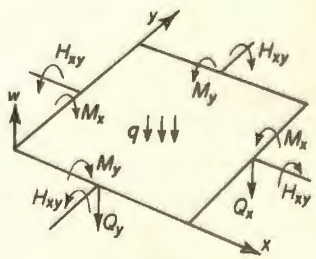
$$K\Delta = R - \beta R^3 \qquad (K > 0,\ \beta > 0),$$

where R is the reaction of the pillar, Δ is the downward deflection of the beam at the point of attachment of the beam to the pillar, K and β are constants.

A load P is applied to the beam at the mid-span point. Find the reactions at the supports B and C. One of the two minimum principles (of potential energy and of complementary energy) is easier to apply to this problem. Solve the problem by a variational method with the appropriate minimum principle.



P10.13



P10.14

10.14. Consider a square plate loaded in the manner shown in Fig. P10.14. Derive Euler's equation and the natural boundary conditions for V to be a minimum when $w(x, y)$ is varied.

$$V = U + A,$$

$$U = \frac{D}{2}\int_0^1\int_0^1 [(w_{xx} + w_{yy})^2 - 2(1 - \nu)(w_{xx}w_{yy} - w_{xy}^2)]\,dx\,dy,$$

$$\begin{aligned} A = \int_0^1\int_0^1 qw\,dx\,dy - \int_0^1 \left(M_x \frac{\partial w}{\partial x}\right)\Big|_{x=0}^{x=1} dy + \int_0^1 (Q_x w)\Big|_{x=0}^{x=1} dy \\ + \int_0^1 \left(H_{xy}\frac{\partial w}{\partial x}\right)\Big|_{x=0}^{x=1} dy - \int_0^1 \left(M_y \frac{\partial w}{\partial y}\right)\Big|_{y=0}^{y=1} dx \\ + \int_0^1 (Q_y w)\Big|_{y=0}^{y=1} dx + \int_0^1 \left(H_{xy}\frac{\partial w}{\partial y}\right)\Big|_{y=0}^{y=1} dx. \end{aligned}$$

11

HAMILTON'S PRINCIPLE, WAVE PROPAGATION, APPLICATIONS OF GENERALIZED COORDINATES

In dynamics, the counterpart of the minimum potential energy theorem is Hamilton's principle. In this chapter we shall discuss this important principle and its applications to vibrations and wave propagations in beams.

Toward the end of the chapter a brief discussion of the so-called *direct methods* of solving variational problems is given. The basic idea is to apply the concept of generalized coordinates to obtain approximate solutions for a continuous system by reducing it to one with a finite number of degrees of freedom. Several important methods—those of Euler, Rayleigh-Ritz-Galerkin, and Kantrovich—will be outlined. The question of convergence, however, must be referred to mathematical treatises. See Bibliography 11.3 on p. 498.

11.1. HAMILTON'S PRINCIPLE

For an oscillating body, with displacements u_i so small that the acceleration is given by $\partial^2 u_i/\partial t^2$ in Eulerian coordinates (Sec. 5.2) the equation of small motion is

$$\text{(1)} \qquad \sigma_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2},$$

where ρ is the density of the material and F_i is the body force per unit volume. Let us again consider virtual displacements δu_i as specified in Sec. 10.7, but instead of a body in static equilibrium we now consider a vibrating body. The variations δu_i must vanish over the boundary surface S_u, where values of displacements are prescribed; but are arbitrary, triply differentiable over the domain V; and are arbitrary also over the rest of the boundary surface S_σ, where surface tractions are prescribed.